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How to build Hamiltonians that transport noncommuting charges in quantum thermodynamics

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Noncommuting conserved quantities have recently launched a subfield of quantum thermodynamics. In conventional thermodynamics, a system of interest and an environment exchange quantities—energy, particles, electric charge, etc.—that are globally conserved and are represented by Hermitian operators. These operators were implicitly assumed to commute with each other, until a few years ago. Freeing the operators to fail to commute has enabled many theoretical discoveries—about reference frames, entropy production, resource-theory models, etc. Little work has bridged these results from abstract theory to experimental reality. This paper provides a methodology for building this bridge systematically: we present a prescription for constructing Hamiltonians that conserve noncommuting quantities globally while transporting the quantities locally. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. Our Hamiltonians may be realized physically with superconducting qudits, with ultracold atoms, and with trapped ions.

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INTRODUCTION

One of thermodynamics' most fundamental and ubiquitous interactions is the exchange of quantities between a system of interest and an environment. Example quantities include energy, particles, and electric charge. As the quantities are conserved globally, we call them 'charges.' (We call even the local quantities 'charges' for convenience, even though the quantities are not conserved locally). Such exchanges happen, for example, in electrochemical batteries, in a cooling cup of coffee, and when a few spins flip to align with a magnetic field. Given such exchanges' pervasiveness, studying their quantum facets is essential for (i) developing the field of quantum thermodynamics[1,2] and (ii) discovering nonclassical features of quantum many-body thermalization in condensed matter; atomic, molecular and optical (AMO) physics; high-energy physics; and chemistry. One important quantum phenomenon is operators' failure to commute with each other: Noncommutation underlies uncertainty relations, measurement disturbance, and more. Therefore, studying exchanges of noncommuting charges is crucial for understanding quantum thermodynamics. As a result, noncommuting charges have been enjoying a heyday[3–29] in quantum-information-theoretic (QIT) thermodynamics.

Lifting the assumption that exchanged charges commute[3–5,13,14,27,30,31] has led to discoveries of truly quantum thermodynamics. Example discoveries include a generalization of the microcanonical state[5], resource theories[3–8,32], a generalization of the majorization preorder[9], a reduction of entropy production by charges' noncommutation[10], and reference-frame designs[11,12]. These discoveries and others have turned noncommuting thermodynamic charges into a growing subfield.

Most of the discoveries have, until recently, belonged to QIT thermodynamics. However, given their fundamental and nonclassical nature, exchanges of thermodynamic noncommuting charges call for bridges to experiments and to many-body physics. Building these bridges requires Hamiltonians that transport noncommuting observables locally while conserving them globally: as stated in the quantum-thermodynamics review[1], 'an abstract view of dynamics, minimal in the details of Hamiltonians, is often employed in quantum information' and so in QIT thermodynamics. In contrast, experiments, simulations, and many-body theory require microscopic Hamiltonians.

Before the present work, it was unknown (i) whether Hamiltonians that transport noncommuting observables locally, while conserving them globally, exist; (ii) how such Hamiltonians look, if they exist; (iii) how to construct such Hamiltonians for given noncommuting charges; and (iv) for which charges such Hamiltonians can be constructed. We answer these questions, enabling the system-and-environment exchange of noncommuting charges to progress from its QIT-thermodynamic birthplace to many-body physics and experiments. Example predictions that merit experimental exploration include (i) the emergence of the quantum equilibrium state in refs. [3–5], (ii) the decrease in entropy production by noncommuting charges[10], (iii) applications of the entropy decrease to quantum engines[33], (iv) the conjecture that noncommuting charges hinder thermalization[14], and (v) the conjecture's application to quantum memories. We open the door to experiments by prescribing how to construct the needed Hamiltonians. Our construction also enables the generalization, to noncommuting charges, of many-body–thermalization tools in condensed matter, AMO physics, and high-energy. Examples include the eigenstate thermalization hypothesis, out-of-time-ordered correlators, and random unitary circuits (e.g., refs. [34–42]).

This paper introduces a prescription for constructing Hamiltonians that overtly move noncommuting charges between subsystems while conserving the charges globally. The charges form a finite-dimensional semisimple complex Lie algebra. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. The prescription also

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produces a convenient basis for the algebra—a basis of charges explicitly transported locally, and conserved globally, by the Hamiltonian. The prescription is general, being independent of any physical platforms. Consequently, the Hamiltonians can be realized with diverse physical systems, such as superconducting circuits, ultracold atoms, and trapped ions.

In a special case, the charges form the Lie algebra $\mathfrak{su}(D)$, N identical subsystems form the global system, and each subsystem corresponds to the Hilbert space $\mathbb{C}^D$. In this example the Schur-Weyl duality describes the Hamiltonians' forms[43,44]: Let the global system (formed from the system of interest and the environment) be many copies of the system of interest. The Hamiltonians are the linear combinations of the permutations of the copies. (Hamiltonians have also been engineered to have SU(D) symmetry without regard to whether noncommuting charges are transported[45,46]). Our results are more general than the Schur-Weyl duality and elucidate the dynamics' physical interpretation. First, our prescription governs a much wider class of algebras: all finite-dimensional, semisimple Lie algebras in which the Killing form induces a metric. Many physically significant algebras satisfy these assumptions—for example, the simple Lie algebras, which include $\mathfrak{su}(D)$. Second, our results are not restricted to systems whose Hilbert spaces are $\mathbb{C}^D$. Finally, the Hamiltonian form specified by the Schur-Weyl duality—a linear combination of permutations—is an abstract construct. How to implement an arbitrary linear combination of permutations is not obvious. In contrast, our Hamiltonians have a clear physical interpretation, manifestly transporting noncommuting charges between subsystems. To our knowledge, no other class of Hamiltonians that transport charges locally and conserve them globally, comparably general to our class, is known.

This paper begins with our setup, detailed in the below section. The section 'Pedagogical explanation' introduces the Hamiltonian-construction prescription pedagogically. We also review mathematical background and illustrate the prescription with an example familiar in quantum information, the Lie algebra $\mathfrak{su}(2)$. The section 'Prescription for constructing the Hamiltonians' synopsizes the prescription, crystallizing the main result, and presents two properties of the prescription. A richer example provides intuition in the section 'su(3) example': Hamiltonians that transport and conserve charges in the Lie algebra $\mathfrak{su}(3)$. The section 'Discussion' concludes with potential realizations of our Hamiltonians in condensed matter, AMO, and high-energy and nuclear physics.

RESULTS

Setup

Consider a global closed quantum many-body system, as in recent thermalization experiments[47–58]. As in conventional statistical mechanics, the global system is an ensemble of N identical subsystems. (We use the term 'ensemble' in the traditional sense of statistical physics: a collection of many identical copies of a system of interest. Such ensembles are often invoked to determine equilibrium probability distributions[59], p. 62). A few of the subsystems form the system of interest; and the rest, an effective environment. Each subsystem corresponds to a Hilbert space $\mathcal{H}$ of finite dimensionality d.

We will construct global Hamiltonians, H^{tot}, that conserve extensive charges defined as follows. Let Q_α denote a Hermitian operator defined on $\mathcal{H}$. We denote by $Q_\alpha^{(j)}$ the observable defined on the j^{th} subsystem's $\mathcal{H}$. Each global observable

$$Q_\alpha^{\text{tot}} := \sum_{j=1}^{N} Q_\alpha^{(j)} \equiv \sum_{j=1}^{N} \mathbb{1}^{\otimes(j-1)} \otimes Q_\alpha^{(j)} \otimes \mathbb{1}^{\otimes(N-j)} \quad (1)$$

will be conserved by design:

$$[H^{\text{tot}}, Q_\alpha^{\text{tot}}] = 0. \quad (2)$$

Although the local $Q_\alpha^{(j)}$ are not conserved, we will sometimes call them, and the Q_α, 'charges' for convenience. One might know, initially, of only c' charges' existence.

These c' Q_α's generate a complex Lie algebra $\mathcal{A}$, which we assume to be finite-dimensional. $\mathcal{A}$ consists of all the charges (as well as non-Hermitian operators, which we ignore). Lie algebras describe many conserved physical quantities: particle number, angular momentum, electric charge, color charge, weak isospin, and our space-time's metric[44,60,61]. We focus on non-Abelian Lie algebras, motivated by quantum thermodynamics that highlights noncommutation: the commutator exemplifies the Lie bracket, $[Q_\alpha, Q_\beta]$.

We assume four more properties of the algebra, to facilitate our proofs. $\mathcal{A}$ is finite-dimensional and semisimple. Representing an observable, $\mathcal{A}$ is over the complex numbers. Also, on $\mathcal{A}$ is defined a Killing form (reviewed below) that induces a metric. Many physically significant algebras satisfy these assumptions—for example, the simple Lie algebras (see the Supplementary Note 1 and refs. [44,60,61]).

Pedagogical explanation

This section describes the prescription for constructing Hamiltonians H^{tot} that conserve noncommuting charges globally [Eq. (2)] while transporting them locally:

$$\left[H^{\text{tot}}, Q_\alpha^{(j)}\right] \neq 0 \quad (3)$$

for some site j. (In every such commutator throughout this paper, one argument implicitly contains tensor factors of $\mathbb{1}$, so that both arguments operate on the same Hilbert space.) We construct two-body interaction terms, then combine them into many-body terms. This explanation provides a pedagogical introduction; the prescription is synopsized in the section 'Prescription for constructing the Hamiltonians'. Here, we illustrate each step with an algebra familiar in quantum information, $\mathfrak{su}(2)$, which describes spin-1/2 angular momentum.

Table 1 lists the simple Lie algebras. Every Cartesian product of simple Lie algebras yields a semisimple Lie algebra $\mathcal{A}$. Such an algebra generates a semisimple Lie group $\mathcal{G}$. For example, if $\mathcal{A}$ consists of angular momentum, $\mathcal{A} = \mathfrak{su}(D)$. The corresponding $\mathcal{G}$ consists of rotations: $\mathcal{G} = \text{SU}(D)$.

An algebra has two relevant properties, a dimension and a rank (Table 1). The dimension, c, equals the number of generators in a basis for the algebra.(We chose the notation c to evoke the c introduced in[14]. There, c was defined as the number of charges. As explained in the section 'Setup' under Results, those charges would form a Lie algebra. Infinitely many charges would therefore exist, the c in[14] would equal infinity, and results in[14] would be

Table 1. Simple Lie algebras: c denotes an algebra's dimension, and r denotes the rank.

Algebra	Dimension (c)	Rank (r)	c/r
$\mathfrak{so}(2D)$	$D(2D-1)$	D	$2D-1$
$\mathfrak{sl}(D+1)$	$(D+1)^2-1$	D	$D+2$
$\mathfrak{so}(2D+1)$	$D(2D+1)$	D	$2D+1$
$\mathfrak{sp}(2D)$	$D(2D+1)$	D	$2D+1$
$\mathfrak{g}_2$	14	2	7
$\mathfrak{f}_4$	52	4	13
$\mathfrak{e}_6$	78	6	13
$\mathfrak{e}_7$	133	7	19
$\mathfrak{e}_8$	248	8	31

We implicitly omit $\mathfrak{so}(2)$ and $\mathfrak{so}(4)$, which are not simple[61]. Also $\mathfrak{su}(D)$ is a simple Lie algebra. However, including $\mathfrak{su}(D)$ would be redundant: the complexification of $\mathfrak{su}(D)$ is isomorphic to $\mathfrak{sl}(D)$.

impractical. We therefore define c as the Lie algebra's finite dimension.) For example, $\mathfrak{su}(2)$ has the Pauli-operator basis $\{\sigma_x, \sigma_y, \sigma_z\}$ and so has a dimension $c = 3$. The rank, r, has a significance that we will encounter shortly.

A representation of $\mathcal{A}$ is a Lie-bracket-preserving map from $\mathcal{A}$ to a set of linear transformations. The adjoint representation maps from $\mathcal{A}$ to linear transformations defined on $\mathcal{A}$. If $x \in \mathcal{A}$, the adjoint representation $\text{ad}(x)$ acts on $y \in \mathcal{A}$ as $\text{ad}(x)(y) := [x, y]$. The adjoint representation features in the Killing form, which we review now. The definition of $\mathcal{A}$ involves a vector space V defined over a field F. A map $V \times V \to F$ is a form. The Killing form is the symmetric bilinear form

$$(x, y) := \text{Tr}(\text{ad}(x)\text{ad}(y)). \tag{4}$$

We say that x and y are Killing-orthogonal if $(x, y) = 0$. We say that subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are Killing-orthogonal if, for all $x \in \mathcal{A}_1$ and $y \in \mathcal{A}_2$, $(x, y) = 0$. We will use the Killing form to construct the preferred basis of charges for $\mathcal{A}$.

Our construction begins with another basis: every finite-dimensional semisimple complex Lie algebra $\mathcal{A}$ has a Cartan–Weyl basis. In fact, $\mathcal{A}$ has infinitely many. Convention may distinguish one Cartan-Weyl basis. We use the conventional $\mathfrak{su}(2)$ basis for concreteness. We use this basis, in our example, for concreteness. In general, one selects an arbitrary Cartan-Weyl basis. The basis contains generators of two types: Hermitian operators and ladder operators.

The number of Hermitian operators is the algebra's rank, r. These operators commute with each other. If $r > 1$, we rescale the operators to endow them with unit Hilbert-Schmidt norms:

$$\text{Tr}(Q_\alpha^\dagger Q_\alpha) = 1. \tag{5}$$

We include these operators, $Q_{\alpha=1,2,\ldots,r}$, in our preferred basis. In the $\mathfrak{su}(2)$ example, $r = 1$; and $Q_1 = \sigma_z$, whose eigenstates $|\pm z\rangle$ correspond to the eigenvalues ± 1. The Q_α's generate a subalgebra, a Cartan subalgebra.

The Cartan-Weyl basis contains, as well as Hermitian operators, ladder operators. They form pairs $L_{\pm\beta}$, for $\beta = 1, 2, \ldots, \frac{c-r}{2}$: Since the Cartan-Weyl basis has c elements, and r of them are Hermitian, there are $c - r$ ladder operators. Each β corresponds to two ladder operators, one raising ($+\beta$) and one lowering ($-\beta$). Hence β runs from 1 to $\frac{c-r}{2}$. Each $L_{\pm\beta}$ raises or lowers at least one Q_α. In the $\mathfrak{su}(2)$ example, the ladder operators $\sigma_{\pm z} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ raise and lower σ_z: $L_{\pm z}|\mp z\rangle = |\pm z\rangle$. In other algebras, an $L_{\pm\beta}$ can raise and/or lower multiple Q_α's. Examples include $\mathfrak{su}(3)$ (explained in the section 'su(3) example').

From each ladder-operator pair, we construct an interaction that couples subsystems j and j'. Let $J_\beta^{(j,j')}$ denote a hopping frequency. An interaction that transports all the charges between j and j', while conserving each charge globally, has the form

$$H^{(j,j')} \propto \sum_{\beta=1}^{(c-r)/2} J_\beta^{(j,j')} \left(L_{+\beta}^{(j)} L_{-\beta}^{(j')} + L_{-\beta}^{(j)} L_{+\beta}^{(j')} \right). \tag{6}$$

We assemble the other terms in $H^{(j,j')}$ from other Cartan-Weyl bases, constructed as follows. Let U denote a general element of the group $\mathcal{G}$. We conjugate, with U, each element of our first Cartan-Weyl basis: For $\alpha = 1, 2, \ldots, r$ and $\beta = 1, 2, \ldots, \frac{c-r}{2}$,

$$Q_\alpha \mapsto U^\dagger Q_\alpha U = Q_{\alpha+r}, \quad \text{and} \tag{7}$$

$$L_{\pm\beta} \mapsto U^\dagger L_{\pm\beta} U = L_{\pm\left(\beta + \frac{c-r}{2}\right)}. \tag{8}$$

We include the new Q_α's (for which $\alpha = r + 1, r + 2, \ldots, 2r$) in our preferred basis for the algebra.

We constrain U such that each new Q_α is Killing-orthogonal to (i) each other new charge Q_β and (ii) each original charge Q_γ:

$$(Q_\alpha, Q_\beta) = (Q_\alpha, Q_\gamma) = 0 \tag{9}$$

for all $\alpha, \beta = r + 1, r + 2, \ldots, 2r$ and all $\gamma = 1, 2, \ldots, r$. This orthogonality restricts U, though not completely. The new Q_α's generate a Cartan subalgebra Killing-orthogonal to the original Cartan subalgebra. The new ladder operators contribute to the interaction:

$$H^{(j,j')} \propto \sum_{\beta=1}^{c-r} J_\beta^{(j,j')} \left(L_{+\beta}^{(j)} L_{-\beta}^{(j')} + \text{h.c.} \right). \tag{10}$$

In the $\mathfrak{su}(2)$ example, U can be represented by $\begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}$, wherein $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. The prescription restricts U only via the Killing-orthogonality of $U^\dagger \sigma_z U$ to U. We enforce only this restriction in the Supplementary Note 2. Here, we choose a U for pedagogical simplicitly: $U = (\mathbb{1} + i\sigma_y)/\sqrt{2}$, such that $Q_{\alpha+r} = Q_2 = \sigma_x$. The new ladder operators, $\sigma_{\pm x} := \left(\frac{\mathbb{1}+i\sigma_y}{\sqrt{2}}\right)\sigma_{\pm z}\left(\frac{\mathbb{1}+i\sigma_y}{\sqrt{2}}\right)$, create and annihilate quanta of the x-component of the angular momentum. The interaction becomes

$$H^{(j,j')} \propto \sum_{\beta=z,x} J_\beta^{(j,j')} \left(\sigma_{+\beta}^{(j)} \sigma_{-\beta}^{(j')} + \text{h.c.} \right). \tag{11}$$

We repeat the foregoing steps: write out the form of a general $U \in \mathcal{G}$. Conjugate each element of the original Cartan-Weyl basis with U. Constrain U such that the new Q_α's are orthogonal to each other and to the older Q_α's. Include the new Q_α's in our preferred basis for the algebra. Form a term, in $H^{(j,j')}$, from the new ladder operators $L_{\pm\beta}$.

Each Cartan-Weyl basis contributes r elements Q_α to the preferred basis. The basis contains c elements, so we form c/r mutually orthogonal Cartan-Weyl bases. c/r equals an integer for the finite-dimensional semisimple complex Lie algebras, according to Proposition 1 in the section 'Prescription for constructing the Hamiltonians'. Table 1 confirms the claim for the simple Lie algebras. Our algebra's finite dimensionality ensures that our prescription halts. The two-body interaction is now

$$H^{(j,j')} = \sum_{\beta=1}^{\frac{c-r}{2}\frac{c}{r}} J_\beta^{(j,j')} \left(L_{+\beta}^{(j)} L_{-\beta}^{(j')} + \text{h.c.} \right). \tag{12}$$

Why is the preferred basis $\{Q_\alpha\}$ preferable? First, the basis endows the Hamiltonian with a simple physical interpretation: $H^{(j,j')}$ transports all these charges locally while conserving them globally. Second, the basis is (Killing-)orthogonal.

In the $\mathfrak{su}(2)$ example, $c/r = 3/1 = 3$. Hence we construct three Cartan-Weyl bases, using two SU(2) elements. If the first unitary was $(\mathbb{1} + i\sigma_y)/\sqrt{2}$, the second unitary is $(\mathbb{1} - i\sigma_x + i\sigma_y + i\sigma_z)/2$, to within a global phase. Consequently, $Q_3 = \sigma_y$, the preferred basis for $\mathcal{A}$ is $\{\sigma_z, \sigma_x, \sigma_y\}$, and

$$H^{(j,j')} = \sum_{\beta=x,y,z} J_\beta^{(j,j')} \left(\sigma_{+\beta}^{(j)} \sigma_{-\beta}^{(j')} + \text{h.c.} \right). \tag{13}$$

Next, we constrain the interaction to conserve every global charge:

$$[H^{(j,j')}, Q_\alpha^{\text{tot}}] = 0 \quad \forall \alpha = 1, 2, \ldots, c. \tag{14}$$

The commutation relations (14) constrain the hopping frequencies $J_\alpha^{(j,j')}$. The frequencies must equal each other in the $\mathfrak{su}(2)$ example: $J_\alpha^{(j,j')} \equiv J^{(j,j')}$ for all α. The Hamiltonian simplifies to[19]

$$H^{(j,j')} = J^{(j,j')} \vec{\sigma}^{(j)} \cdot \vec{\sigma}^{(j')}. \tag{15}$$

This Heisenberg model is known to have SU(2) symmetry and so to conserve each global spin component $\sigma_\alpha^{\text{tot}} := \sum_{j=1}^N \sigma_\alpha^{(j)}$. But the Hamiltonian is typically written in the dot-product form (15), as

$$H^{(j,j')} \propto \sum_{\alpha=x,y,z} \sigma_\alpha^{(j)} \sigma_\alpha^{(j')}. \tag{16}$$

or in the z-biased form $H^{(j,j')} \propto 2(\sigma_{+z}^{(j)}\sigma_{-z}^{(j')} + \sigma_{-z}^{(j)}\sigma_{+z}^{(j')}) + \sigma_z^{(j)}\sigma_z^{(j')}$. None of these three forms reveals that the Heisenberg model transports noncommuting charges between subsystems. Our expression (13) and our prescription do. In relativistic field theories, making the action manifestly Lorentz-invariant is worthwhile; analogously, making the Hamiltonian manifestly transport noncommuting charges locally, while conserving them globally, is worthwhile. Furthermore, our prescription constructs Hamiltonians that overtly transport noncommuting charges locally and conserve the charges globally not only in this simple $\mathfrak{su}(2)$ example, but also for all finite-dimensional semisimple complex Lie algebras on which the Killing form induces a metric—including algebras for which this prescription does not produce the Heisenberg Hamiltonian. Supplementary Note 3 discusses a generalization of the simple form (15).

We have constructed a two-body interaction $H^{(j,j')}$ that couples subsystems j and j'. We construct k-body terms $H^{(j,j',\ldots,j^{(k)})}$ by multiplying two-body terms (12) together, constraining the couplings such that $[H^{(j,j',\ldots,j^{(k)})}, Q_\alpha^{\mathrm{tot}}] = 0$, and subtracting off any fewer-body terms that appear in the product. Section II C details the formalism. In the $\mathfrak{su}(2)$ example, a three-body interaction has the form (see Supplementary Note 2)

$$H^{(j,j',j'')} \propto H^{(j,j')}H^{(j',j'')}H^{(j'',j)} \tag{17}$$

$$\propto J^{(j,j',j'')}\big[(\sigma_x\sigma_y\sigma_z + \sigma_y\sigma_z\sigma_x + \sigma_z\sigma_x\sigma_y) - (\sigma_z\sigma_y\sigma_x + \sigma_x\sigma_z\sigma_y + \sigma_y\sigma_x\sigma_z)\big]. \tag{18}$$

wherein $J^{(j,j',j'')} \in \mathbb{R}$.

The Hamiltonian we constructed may be integrable. For example, the one-dimensional (1D) nearest-neighbor Heisenberg model is integrable[62]. Integrable Hamiltonians have featured in studies of noncommuting charges in thermodynamics[21]. But one might wish for the system to thermalize as much as possible, as is promoted by nonintegrability[37,63]. Geometrically nonlocal couplings, many-body interactions, and multidimensional lattices tend to break integrability. Hence one can add terms $H^{(j,j')}$ and $H^{(j,j',\ldots,j^{(k)})}$ to the global Hamiltonian H^{tot}, and keep growing the lattice's dimensionality, until H^{tot} becomes nonintegrable. Nonintegrability may be diagnosed with, e.g., energy-gap statistics[37]. In the $\mathfrak{su}(2)$ example, one can break integrability by creating next-nearest-neighbor couplings or by making the global system two-dimensional[19].

Prescription for constructing the Hamiltonians

Here, we synopsize the prescription elaborated on in the section 'Pedagogical explanation'. Then, we present two results pertinent to the prescription. We construct, as follows, Hamiltonians that transport noncommuting charges locally and conserve the charges globally:

1. Identify an arbitrary Cartan-Weyl basis for the algebra, $\mathcal{A}$.
2. The Cartan-Weyl basis contains r Hermitian operators that commute with each other. Scale each such operator such that it has a unit Hilbert-Schmidt norm [Eq. (5)]. Label the results $Q_{\alpha=1,2,\ldots,r}$. Include them in the preferred basis for the algebra.
3. The other Cartan-Weyl-basis elements are ladder operators that form raising-and-lowering pairs: $L_{\pm\beta}$, for $\beta = 1, 2, \ldots c - r$. From each pair, form one term in the two-body interaction, $H^{(j,j')}$ [Eq. (6)].
4. Write out the form of the most general element $U \in \mathcal{G}$ of the Lie group $\mathcal{G}$ generated by $\mathcal{A}$. Conjugate each charge Q_α and each ladder operator $L_{\pm\beta}$ with U [Eq. (7)]. The new charges and new ladder operators, together, form another Cartan–Weyl basis.
5. Constrain U such that every new charge Q_α is Killing-orthogonal to (i) each other new charge and (ii) each charge already in the basis [Eq. (9)].
6. Include each new Q_α in the basis for $\mathcal{A}$.
7. From each new pair $L_{\pm\beta}$ of ladder operators, form a term in the two-body interaction $H^{(j,j')}$ [Eq. (10)].
8. Repeat steps 4-7 until having identified c/r Cartan-Weyl bases, wherein c denotes the algebra's dimension. Each Cartan-Weyl basis contributes r elements Q_α to the preferred basis for $\mathcal{A}$. The basis is complete, containing $r \cdot \frac{c}{r} = c$ elements.
9. Constrain the two-body interaction to conserve each global charge [Eq. (14)], for all $\alpha = 1, 2, \ldots, c$. Solve for the frequencies $J_\beta^{(j,j')}$ that satisfy this constraint.
10. If a k-body interaction is desired, for any $k > 2$: Perform the following substeps for $\ell = 3, 4, \ldots, k$: Multiply together ℓ unconstrained two-body interactions (12) cyclically:
$$H^{(j,j',\ldots,j^{(\ell)})} = H^{(j,j')}H^{(j',j'')}\ldots H^{(j^{(\ell-1)},j^{(\ell)})} \times H^{(j^{(\ell)},j)}. \tag{19}$$
Constrain the couplings so that $[H^{(j,j',\ldots,j^{(\ell)})}, Q_\alpha^{\mathrm{tot}}] = 0$ for all α. If $H^{(j,j',\ldots,j^{(\ell)})}$ contains fewer-body terms that conserve all the Q_α^{tot}, subtract those terms off.
11. Sum the accumulated interactions $H^{(j,j',\ldots,j^{(k)})}$ over the subsystems $j, j', \ldots$ to form H^{tot}.
12. If H^{tot} is to be nonintegrable, add longer-range interactions and/or large-kk-body interactions until breaking integrability, as signaled by, e.g., energy-gap statistics.

Having synopsized our prescription, we present two properties of it. The first property ensures that the prescription runs for an integer number of iterations (step 8).

Proposition 1. Consider any finite-dimensional semisimple complex Lie algebra. The algebra's dimension, c, and rank, r, form an integer ratio: $c/r \in \mathbb{Z}_{>0}$.

We prove this proposition in the Supplementary Note 4. The second property characterizes the prescription's output.

Theorem 1. The charges $Q_1, Q_2, \ldots, Q_c$ produced by the prescription form a basis for the algebra $\mathcal{A}$.

Proof. The charges are Killing-orthogonal by construction: $(Q_\alpha, Q_\beta) = 0$ for all α, β. The Killing form induces a metric on $\mathcal{A}$ by assumption. Therefore, the Q_α are linearly independent according to this metric.. The prescription produces c charges (step 8). c denotes the algebra's dimension, the number of elements in each basis for $\mathcal{A}$. Hence every linearly independent set of $c\mathcal{A}$ elements forms a basis for $\mathcal{A}$. Hence the Q_α form a basis.

$su(3)$ example

Section II B illustrated the Hamiltonian-construction prescription with the algebra $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$ example offered simplicity but lacks other algebras' richness: In other algebras, each Cartan–Weyl basis contains multiple Hermitian operators and multiple ladder-operator pairs. We demonstrate how our prescription accommodates this richness, by constructing a two-body Hamiltonian that transports $\mathfrak{su}(3)$ elements locally while conserving them globally. Such Hamiltonians may be engineered for superconducting qutrits, as sketched in Discussion. However, this $\mathfrak{su}(3)$ example only illustrates our more general prescription, which works for all finite-dimensional semisimple complex Lie algebras on which the Killing form induces a metric.

Each basis for $\mathfrak{su}(3)$ contains $c = 8$ elements. The most famous basis consists of the Gell-mann matrices, $\lambda_{k=1,2,\ldots,8}$[64]. The λ_k generalize the Pauli matrices in certain ways, being traceless and Killing-orthogonal. From the Gell-mann matrices is constructed the conventional Cartan-Weyl basis[65], reviewed in the Supplementary Note 5. The $r = 2$ Hermitian elements are Gell-mann matrices:

$$Q_1 = \lambda_3, \quad \text{and} \quad Q_2 = \lambda_8. \tag{20}$$

Q_1 and Q_2 belong in the preferred basis of charges for $\mathfrak{su}(3)$. For pedagogical clarity, we will identify all the charges before addressing the ladder operators.

A general element $U \in SU(3)$ contains eight real parameters. In the Euler parameterization[66],

$$U = e^{i\lambda_3\phi_1/2}e^{i\lambda_2\phi_2/2}e^{i\lambda_3\phi_3/2}e^{i\lambda_5\phi_4/2} \times e^{i\lambda_3\phi_5/2}e^{i\lambda_2\phi_6/2}e^{i\lambda_3\phi_7/2}e^{i\lambda_8\phi_8/2}. \tag{21}$$

The parameters $\phi_1, \phi_3, \phi_5, \phi_7 \in [0, 2\pi)$; $\phi_2, \phi_4, \phi_6 \in [0, \pi]$; and $\phi_8 \in [0, 2\sqrt{3}\pi)$. We now constrain U, identifying the instances U_{i} that map the first charges to $Q_3 = U_{\text{i}}^\dagger Q_1 U_{\text{i}}$ and $Q_4 = U_{\text{ii}}^\dagger Q_2 U_{\text{ii}}$ that are Killing-orthogonal to each other and to the original charges. Supplementary Note 5 contains the details. We label with a superscript (i) the parameters used to fix U_{i}: $\phi_1^{(\text{i})}$, $\phi_3^{(\text{i})}$, $\phi_7^{(\text{i})}$, $\phi_8^{(\text{i})}$, and $n^{(\text{i})}$. For convenience, we package several parameters together: $a^{(\text{i})} := \frac{1}{2}\left(\phi_3^{(\text{i})} - \phi_7^{(\text{i})} - \sqrt{3}\phi_8^{(\text{i})} + \pi n^{(\text{i})} + \frac{\pi}{2}\right)$, and $b^{(\text{i})} := a^{(\text{i})} + \phi_7^{(\text{i})}$. In terms of these parameters, the new charges have the forms (see Supplementary Note 5)

$$Q_3 = \frac{1}{\sqrt{3}}\Big[(-1)^{n^{(\text{i})}+1}\sin(a^{(\text{i})} - b^{(\text{i})})\lambda_1 - (-1)^{n^{(\text{i})}}\cos(a^{(\text{i})} - b^{(\text{i})})\lambda_2 - \sin(a^{(\text{i})})\lambda_4 - \cos(a^{(\text{i})})\lambda_5 + \sin(b^{(\text{i})})\lambda_6 + \cos(b^{(\text{i})})\lambda_7\Big] \quad \text{and} \tag{22}$$

$$Q_4 = \frac{(-1)^{n^{(\text{i})}}}{\sqrt{3}}\Big[(-1)^{n^{(\text{i})}+1}\cos(a^{(\text{i})} - b^{(\text{i})})\lambda_1 + (-1)^{n^{(\text{i})}}\sin(a^{(\text{i})} - b^{(\text{i})})\lambda_2 + \cos(a^{(\text{i})})\lambda_4 - \sin(a^{(\text{i})})\lambda_5 + \cos(b^{(\text{i})})\lambda_6 - \sin(b^{(\text{i})})\lambda_7\Big]. \tag{23}$$

Q_3 has the same form as Q_5 and Q_7, which satisfy the same Killing-orthogonality conditions. Similarly, Q_4 has the same form as Q_6 and Q_8. The later charges' parameters $a^{(\ell)}$ and $b^{(\ell)}$ are more restricted, however (see Supplementary Note 5). We have identified our preferred basis of charges.

Let us construct the ladder operators and Hamiltonian. Each Cartan–Weyl basis contains $c - r = 8 - 2 = 6$ ladder operators. The conventional Cartan–Weyl basis contains ladder operators formed from Gell-man matrices:

$$L_{\pm 1} := \tfrac{1}{2}(\lambda_1 \pm i\lambda_2), \quad L_{\pm 2} := \tfrac{1}{2}(\lambda_4 \pm i\lambda_5), \quad \text{and} \quad L_{\pm 3} := \tfrac{1}{2}(\lambda_6 \pm i\lambda_7). \tag{24}$$

Transforming these operators with unitaries $U_{\text{ii,iii,iv}}$ yields $L_{\pm 4}$ through $L_{\pm 12}$, whose forms appear in the Supplementary Note 5. From each ladder operator, we form one term in the two-body Hamiltonian (6).

Finally, we determine the hopping frequencies $J_\alpha^{(j,j')}$, demanding that $[H^{(j,j')}, Q_\alpha^{\text{tot}}] = 0$ for all α. For all possible values of the $a^{(\ell)}$, $b^{(\ell)}$, and $n^{(\ell)}$, if all the frequencies are nonzero, then all the frequencies equal each other. We set $J_\alpha^{(j,j')} \equiv \frac{4}{3}\, J^{(j,j')}$, such that

$$H^{(j,j')} = J^{(j,j')}\sum_{\alpha=1}^{8}\lambda_\alpha^{(j)}\lambda_\alpha^{(j')} \propto \sum_{\alpha=1}^{8} Q_\alpha^{(j)}Q_\alpha^{(j')}. \tag{25}$$

The Hamiltonian collapses to a simple form analogous to the $\mathfrak{su}(2)$ example's Eq. (16) (see Supplementary Note 3).

DISCUSSION

We have presented a prescription for constructing Hamiltonians that transport noncommuting charges locally while conserving the charges globally. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. The prescription produces, as well as Hamiltonians, preferred bases of charges that are (i) overtly transported locally and conserved globally and (ii) Killing-form-orthogonal. This construction works whenever the charges form a finite-dimensional semisimple complex Lie algebra on which the Killing form induces a metric. Whether there exists any Hamiltonians that transport charges locally, while conserving the charges globally, outside of those found by our prescription, is an interesting open question for theoretical exploration.

This work provides a systematic means of bridging noncommuting thermodynamic charges from abstract quantum information theory to condensed matter, AMO physics, and high-energy and nuclear physics. The mathematical results that have accrued[3–25] can now be tested experimentally, via our construction. This paper's introduction highlights example results in that merit testing. Such experiments' benefits include the simulation of quantum systems larger than what classical computers can simulate, the uncovering of behaviors not predicted by theory, and the grounding of abstract QIT thermodynamics in physical reality.

In addition to harnessing controlled platforms to study noncommuting charges' quantum thermodynamics, one may leverage that quantum thermodynamics to illuminate high-energy and nuclear physics. Such physics includes non-Abelian gauge theories, such as quantum chromodynamics. How to define and measure such theories' thermalization is unclear[67]. One might gain insights by using our dynamics as a bridge from quantum thermodynamics to non-Abelian field theories.

As mentioned above, the Heisenberg model (13) can be implemented with ultracold atoms and trapped ions[68–73]. Reference[19] details how to harness these setups to study noncommuting thermodynamic charges. We introduce a more complex example here: We illustrate, with superconducting qubits, how today's experimental platforms can implement the $\mathfrak{su}(3)$ instance of our general prescription.

Superconducting circuits can serve as qudits with Hilbert-space dimensionalities $d \geq 2$[74]. Qutrits have been realized with transmons, slightly anharmonic oscillators[75]. The lowest two energy levels often serve as a qubit, but the second energy gap nearly equals the first. Hence the third level can be addressed relatively easily[76]. Superconducting qutrits offer a tabletop platform for transporting and conserving $\mathfrak{su}(3)$ charges as in the section 'su(3) example'.

Experiments with ≤5 qutrits have been run[77,78], Furthermore, many of the tools used to control and measure superconducting qubits can be applied to qutrits[76,79–88]. A noncommuting-charges-in-thermodynamics experiment may begin with preparing the qutrits in an approximate microcanonical subspace, a generalization of the microcanonical subspace that accommodates noncommuting charges[14]. Such a state preparation may be achieved with weak measurements[19], which have been performed on superconducting qudits through cavity quantum electrodynamics[89].

T_2^* relaxation times of ~39 µs, for the lowest energy gap, and ~14 µs, for the second-lowest gap, have been achieved[78]. Meanwhile, two-qutrit gates can be realized in ~$10 - 10^2$ ns[78,90,91]. Some constant number of such gates may implement one three-level gate that simulates a term in our Hamiltonian. If the number is order-10, information should be able to traverse an 8-qutrit system ~10 times before the qutrits decohere detrimentally. According to numerics in Yunger et al.[14], a small subsystem nears thermalization once information has had time to traverse the global system a number of times linear in N. Therefore, realizations of our Hamiltonians are expected to thermalize the system internally. The states of small subsystems, such as qutrit pairs, can be read out via quantum state tomography[76,79–82]. Hence superconducting qutrits, and other platforms, can import noncommuting charges from quantum thermodynamics to many-body physics, by simulating the Hamiltonians constructed here.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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AUTHOR CONTRIBUTIONS

N.Y.H. developed the prescription, managed the project, and led the paper writing. S. M. worked out the su(3) example, proofs, Supplementary Notes, and superconducting-qutrit details, in addition to leading the referee revisions.

COMPETING INTERESTS

The authors declare no competing interests.

ADDITIONAL INFORMATION

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