

Resource theory of quantum uncomplexity

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Quantum complexity is emerging as a key property of many-body systems, including black holes, topological materials, and early quantum computers. A state’s complexity quantifies the number of computational gates required to prepare the state from a simple tensor product. The greater a state’s distance from maximal complexity, or “uncomplexity,” the more useful the state is as input to a quantum computation. Separately, resource theories—simple models for agents subject to constraints—are burgeoning in quantum information theory. We unite the two domains, confirming Brown and Susskind’s conjecture that a resource theory of uncomplexity can be defined. The allowed operations, *fuzzy operations*, are slightly random implementations of two-qubit gates chosen by an agent. We formalize two operational tasks, uncomplexity extraction and expenditure. Their optimal efficiencies depend on an entropy that we engineer to reflect complexity. We also present two monotones, uncomplexity measures that decline monotonically under fuzzy operations, in certain regimes. This work unleashes on many-body complexity the resource-theory toolkit from quantum information theory.

Quantum complexity has recently swept from quantum computation across many-body physics. A state’s *quantum complexity* quantifies the difficulty of preparing the state from a simple fiducial state. For instance, a random circuit’s output has a quantum complexity that advantages certain quantum computations over their classical counterparts [1–3]. In condensed matter, topological phases are distinguished by complexities that scale linearly with the system size [4, 5]. In many-body physics, chaotic evolutions increase complexity beyond when most physical quantities, such as correlators, equilibrate [6]. This observation underpins a proposal about the anti-de-Sitter-space/conformal-field-theory (AdS/CFT) holographic correspondence: There, a wormhole connecting two black holes is dual to a field-theoretic state. The state’s complexity is conjectured to be proportional to the wormhole’s length [7–12]. Such examples render complexity as a physical quantity that illuminates chaotic systems’ behaviors.

Quantum computation is best begun with a low-complexity state (Fig. 1(a)): A quantum computer needs “clean” qubits in a simple product state, as we need blank paper when computing with a pencil. To quantify a state’s resourcefulness in computation, Brown and Susskind define a state’s *uncomplexity* as the gap between the state’s greatest possible complexity and actual complexity (Fig. 1(a)) [10]. According to a counting ar-

gument, an n -qubit state’s maximum possible complexity, \mathcal{C}_{\max} , scales as e^n [14]. Brown and Susskind argue that uncomplexity decreases monotonically under random, chaotic dynamics; complexity obeys an analog of the second law of thermodynamics [10]. This “second law of uncomplexity” led Brown and Susskind to conjecture that a resource theory for quantum uncomplexity can be defined.

A *resource theory* features an agent able to perform any chosen operation that satisfies simple rules. We study which state transformations these operations can and cannot effect [16]. States impossible to prepare are scarce *resources*, which may facilitate operational tasks. If a resource theory’s rules encode fundamental constraints of Nature, the conclusions extend from the agent’s capabilities to natural evolutions. For example, in the resource theory of entanglement, agents perform only local quantum operations; entanglement between far-apart sites is a resource [17]. Resource theories model diverse phenomena including informational nonequilibrium [18–20], thermodynamics [21–31], coherence [32–40], and quantum channels [41–45]. From their origins in quantum information theory, resource theories have recently extended into other fields of science [46–65]. In this spirit, and motivated by these studies’ impacts on their disciplines, we introduce a resource theory for quantum uncomplexity at the intersection of high-energy theory and condensed matter.

This paper confirms Brown and Susskind’s conjecture that a resource theory for uncomplexity can be defined.

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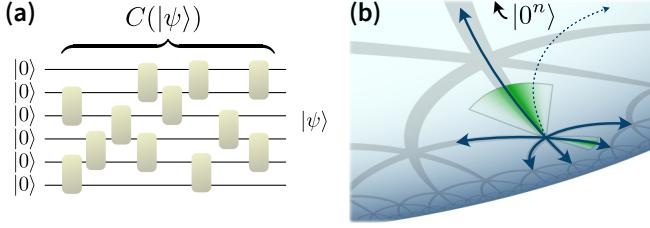


FIG. 1: State complexity and its geometry. **(a)** A pure n -qubit state $|\psi\rangle$ has an *exact complexity* $C(|\psi\rangle)$ equal to the least number of gates required to prepare $|\psi\rangle$ from $|0^n\rangle$. The state's *exact uncomplexity* is the distance $C_{\max} - C(|\psi\rangle)$ to the maximal n -qubit state complexity, $C_{\max} \sim e^n$. **(b)** In a revision of Nielsen's geometry [13, 14], complexity curves the state space negatively [15]. Applying a gate corresponds to moving a unit length along a direction in this geometry. Most directions point toward higher-complexity states. To transform into a generic state $|\psi'\rangle$, $|\psi\rangle$ likely must pass through $|0^n\rangle$; no significantly shorter path exists. Consequently, uncomplexity is desirable in quantum computation: Given a complex $|\psi\rangle$, to prepare a desired output $|\psi'\rangle$, one must uncompute $|\psi'\rangle$ to $|0^n\rangle$, incurring an overhead. Our resource theory introduces randomness (green shaded regions) into the gates. Such gates tend to increase the state's complexity.

Using the resource theory, we define two operational tasks: *Uncomplexity extraction* distills pure qubits from an arbitrary state. *Expenditure* uncomplexity, one can imitate an arbitrary state. We quantify these tasks' optimal efficiencies with an entropy that we introduce. This *complexity entropy* measures the entropy that a state appears to have, to a realistic observer able to measure only simple observables. In certain regimes, we prove, the complexity entropy is a *monotone*, or resource measure, decreasing monotonically under allowed operations. We conjecture that this monotonicity manifests broadly. Furthermore, we prove a version of Brown and Susskind's second law of uncomplexity in the resource theory.

A challenge in defining the resource theory follows from the agent's agency, or ability to choose operations. Designating those operations as gates echoes circuit-based complexity studies. If able to choose any gates, though, the agent can uncompute any pure state to the maximally uncomplex state. Uncomplexity will be easily accessible, not a scarce resource; the resource theory will be a mockery. We rectify this shortcoming by randomizing the gates a little: The agent can perform any slightly random gate, which we call a *fuzzy gate*. Should a state undergo too many fuzzy gates, it grows too random to be useful. The fuzziness ensures that the agent likely will not increase a state's uncomplexity (Fig. 1(b)). This fuzziness echoes the widespread modeling of chaos with randomness [66–68].

This work is organized as follows. We define the resource theory of uncomplexity, then the complexity entropy. The entropy quantifies the optimal efficiencies with which uncomplexity can be extracted and expended,

as formalized and proved next. We identify two fuzzy-operation monotones and conclude with the opportunities unveiled by this work.

Definition of the resource theory of uncomplexity.—We consider a system of n qubits. Denote by σ_z the Pauli z -operator and by $|0\rangle$ the eigenvalue-1 σ_z eigenvector. Let $|0^k\rangle := |0\rangle^{\otimes k}$, and let $\mathbb{1}$ denote the one-qubit identity operator.

Fuzzy gates form the building blocks of the allowed operations. The agent can attempt to perform any 2-qubit gate $U \in \text{SU}(4)$ on any two qubits. Our results generalize to two scenarios in holographic literature [10, 69]:

First, each gate can couple $k \geq 2$ qubits. Second, the target gates U can form a discrete set.

The gate implementation suffers from noise as follows. Denote by $d\tilde{U}$ the Haar (uniform) measure over $\text{SU}(4)$. Fix an error parameter $\epsilon > 0$. Denote by $p_{U,\epsilon}(\tilde{U})$ a normalized probability density, over $\text{SU}(4)$, that satisfies two assumptions: (i) $p_{U,\epsilon}$ introduces noise in all directions of the 2-qubit-gate space around U : $p_{U,\epsilon}$ is nonzero on an open set that contains U . (ii) $p_{U,\epsilon}(\tilde{U})$ sends U no farther than ϵ away: $p_{U,\epsilon}(\tilde{U})$ vanishes for all unitaries \tilde{U} that satisfy $\|U - \tilde{U}\|_\infty > \epsilon$, wherein $\|A\|_\infty$ denotes the operator norm of A . If the agent tries to perform U on two qubits, those qubits undergo a gate \tilde{U} chosen according to $p_{U,\epsilon}(\tilde{U}) d\tilde{U}$.

We can illustrate the density $p_{U,\epsilon}$ with two examples. First, \tilde{U} can be chosen uniformly randomly from the two-qubit unitaries $O(\epsilon)$ -close to U in any norm. Second, denote by $\{P_{j,k}\}$ any basis for the traceless 2-qubit Hermitian operators. Assign random coefficients $\alpha_{j,k} \in [-O(\epsilon), O(\epsilon)]$. The Hamiltonian $H = \sum \alpha_{j,k} P_{j,k}$ perturbs U into $\tilde{U} = e^{iH}U$.

Definition 1 (Fuzzy gates and operations). *Every fuzzy gate is defined in terms of an arbitrarily chosen two-qubit gate U . The fuzzy gate U is selected uniformly randomly according to a distribution $p_{U,\epsilon}(\tilde{U})$ that satisfies conditions (i) and (ii) above. Every composition of fuzzy gates is a fuzzy operation.*

The fuzzy operations are the allowed operations, which define the resource theory. They form a set closed under composition, as required in a resource theory [16]. A fuzzy gate is a unitary sampled according to $p_{U,\epsilon}(\tilde{U}) d\tilde{U}$. Hence we view the evolved state as random and pure, avoiding mixed states as in much holographic, random-circuit, and chaos literature. Alternatively, a fuzzy gate can be represented with the quantum channel $\mathcal{E}(\dots) = \int \tilde{U}(\dots) \tilde{U}^\dagger p_{U,\epsilon}(\tilde{U}) d\tilde{U}$ [70].

No states or partial traces are allowed. This lack is unusual but has precedents in the resource theories of (i) heat and work [71] and (ii) random reversible operations [70]. No states are free because any tensored-on state benefits quantum computation: Consider tensoring a maximally complex m -qubit state onto a maximally complex n -qubit state. C_{\max} grows to $\sim e^{n+m}$, whereas

the actual complexity grows only to $\sim e^n + e^m$. Hence the tensoring-on creates uncomplexity. Even tensoring on a maximally mixed state can boost computational power, as shown by the one-clean-qubit computational model [10, 14, 72].

Complexity entropy.—We introduce an entropy that quantifies tasks’ efficiencies in the resource theory of uncomplexity [10]. For instance, consider an n -site system undergoing a random, chaotic evolution. Let the system occupy a pure state ρ_t at time t . The von Neumann entropy $H_{\text{VN}}(\rho_t) := -\text{Tr}(\rho_t \log_2 \rho_t) = 0 \forall t$, regardless of ρ_t ’s complexity. Furthermore, site j occupies some reduced state $\rho_t^{(j)}$. The entanglement entropy $H_{\text{VN}}(\rho_t^{(j)})$ can saturate in a time $O(n^0)$. In contrast, the complexity can grow for a time $\sim e^n$ [6]. We overcome these obstacles, introducing an entropy suited to complexity, inspired by [73]. Reference [74] will detail the entropy’s properties.

The complexity entropy quantifies how random a state appears if probed only through sufficiently simple observables. For instance, consider measuring a simple observable of a highly complex state $|\psi\rangle$. The outcome is highly random, as if $|\psi\rangle$ were highly entropic [75]. Reference [73] introduces a strategy for quantifying this apparent randomness: Use an operational measure of the state’s distinguishability from the maximally mixed state, and restrict the associated distinguishability task’s complexity. Inspired by this approach, we use the *hypothesis testing entropy* [76–82].

In a hypothesis test, one receives a state, ρ or σ , and guesses which state arrived. The most general approach involves a two-outcome measurement, represented quantum-information-theoretically with a positive-operator-valued measure (POVM) [83] $\{Q, \mathbb{1} - Q\}$, wherein $0 \leq Q \leq \mathbb{1}$. Outcome Q suggests that the state was ρ , and $\mathbb{1} - Q$ suggests that the state was σ . Let $\sigma = \mathbb{1}/2^n$. Suppose that one must, if the state is ρ , guess ρ with probability $\geq \eta \in (0, 1]$. The minimum probability of wrongly guessing $\sigma = \mathbb{1}/2^n$, if the state is ρ , defines the hypothesis-testing entropy,

$$H_h^\eta(\rho) := \log_2 \left(\min_{\substack{0 \leq Q \leq \mathbb{1} \\ \text{Tr}(Q\rho) \geq \eta}} \{\text{Tr}(Q)\} \right). \quad (1)$$

$H_h^\eta(\rho)$ satisfies properties characteristic of entropies [78, 81, 84].

We restrict the measurement’s computational difficulty: First, we define a set M_0 of zero-measurement-complexity measurement operators. Under such an operator’s action, each qubit is (i) projected onto $|0\rangle$ or (ii) not touched (evolved with $\mathbb{1}$). Define the variable α_j as 1 if qubit j is projected and as 0 otherwise. If $(|0\rangle\langle 0|)^0 \equiv \mathbb{1}$, the measurement operator has the form

$$\bigoplus_{j=1}^n (j|0\rangle\langle 0|_j)^{\alpha_j} = Q_0 \in M_0. \quad (2)$$

Fix an integer $r \geq 0$. Consider performing $\leq r$ 2-local gates, effecting a unitary U_r , before measuring a Q_0 . The net effect, we define as a *complexity- r measurement*. The operators $U_r^\dagger Q_0 U_r$ form a set M_r . Restricting to M_r the Q in (1), we define the complexity entropy.

Definition 2 (Complexity entropy). *The complexity entropy of an n -qubit state ρ is, for $\eta \in (0, 1]$,*

$$H_c^{r,\eta}(\rho) := \min_{\substack{Q \in M_r, \\ \text{Tr}(Q\rho) \geq \eta}} \{\log_2(\text{Tr}(Q))\}. \quad (3)$$

We can understand the definition through two extremes. Suppose that $\rho = |\psi\rangle\langle\psi|$ is pure. Let the number r of performable gates be high, compared to the number of gates needed to prepare $|\psi\rangle$. U_r approximately undoes the gates in $|\psi\rangle$, so most tensor factors in (2) should be $|0\rangle\langle 0|$ ’s. Q projects onto a low-dimensional subspace, so $\text{Tr}(Q)$ is small, as the minimization requires. In the extreme case, $Q = |0^n\rangle\langle 0^n|$, and $H_c^{r,\eta}(|\psi\rangle) = 0$. If the projected-onto subspace is so small, is $\text{Tr}(Q|\psi\rangle\langle\psi|) \geq \eta$ violated? No, as $U_r|\psi\rangle \approx |0^n\rangle$ by assumption. Therefore, if $|\psi\rangle$ is uncomplex while many gates are available, $H_c^{r,\eta}(|\psi\rangle) \gtrsim 0$.

Contrariwise, let $|\psi\rangle$ be highly complex and only a few gates be performable (let r be small). Most qubits, probed locally, likely resemble $1/2$. Each $1/2$ halves $\text{Tr}(Q|\psi\rangle\langle\psi|)$ if multiplying a $|0\rangle\langle 0|$ in Q . To satisfy $\text{Tr}(Q|\psi\rangle\langle\psi|) \geq \eta$, Q must contain many 1 ’s. Each 1 doubles $\text{Tr}(Q)$. In the extreme case, $Q = \mathbb{1}^{\otimes n}$, and $H_c^{r,\eta}(|\psi\rangle) = n$. Therefore, if $|\psi\rangle$ is complex while few gates are performable, $H_c^{r,\eta}(|\psi\rangle) \lesssim n$.

We can choose different conventions in Definition 2. First, we can define M_0 in terms of the measurements natural for a given platform. Second, we can define M_r in terms of any complexity measure, such as Nielsen’s [13, 85–87], rather than in terms of r gates.

Uncomplexity extraction and expenditure.—The complexity entropy, we prove, quantifies the optimal efficiencies of two operational tasks that we formalize, using the resource theory (Fig. 2). The allowed operations’ fuzziness limits the number of gates performable before the state grows too random to be useful. Our theorems therefore portray the agent as effecting We quantify states’ closeness with the trace distance, $\mathcal{T}(\rho, \bar{\rho}) := \|\rho - \bar{\rho}\|_1/2$, wherein $\|A\|_1 = \text{Tr}\sqrt{A^\dagger A}$ denotes the trace norm.

We define *uncomplexity extraction* as follows. Let ρ denote any n -qubit state, $r \in \mathbb{Z}_{\geq 0}$, and $\delta \geq 0$. We seek a circuit of $\leq r$ fuzzy gates, and a selection of w qubits, with the following property. Suppose that ρ undergoes the circuit and then the nonselected qubits are discarded. The result is δ -close to $|0^w\rangle$ in trace distance (Fig. 2(a)). The following theorem establishes an extraction protocol’s existence and near-optimality. Appendix A contains the proof.

Theorem 1 (Uncomplexity extraction). *Let ρ , r , and δ be as above, and assume that $\delta \geq r\epsilon$. For every $\eta \in [1 - (\delta - r\epsilon)^2, 1]$, some protocol extracts $w =$*

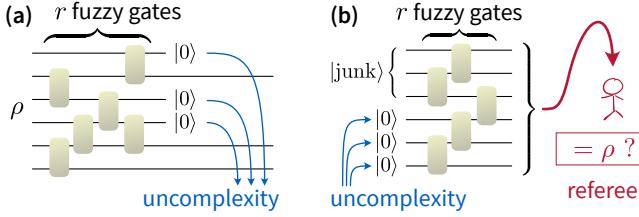


FIG. 2: Operational tasks of uncomplexity extraction and expenditure in the resource theory. Since gates are fuzzy, the agent can perform only $\leq r$ gates, lest the state grow too noisy to be useful. **(a)** Extracting uncomplexity from a state ρ , one applies $\leq r$ fuzzy gates. The number of qubits left in the state $|0\rangle$ is the extractable uncomplexity, which equals the complexity entropy of ρ . **(b)** Given enough $|0\rangle$'s, an agent can “spend” uncomplexity to imitate ρ : The agent performs r fuzzy gates, preparing a state believed, by a computationally bounded referee, to be ρ .

$n - H_c^{r,\eta}(\rho)$ qubits δ -close to $|0^w\rangle$ in trace distance. Conversely, every uncomplexity-extraction protocol obeys $w \leq n - H_c^{r,1-\delta}(\rho)$.

Another bound governs the uncomplexity cost of simulating a state ρ . Suppose that a computationally limited referee, upon receiving an n -qubit state, performs a hypothesis test between ρ and the completely ignorant observer’s null hypothesis, $1/2^n$. Computationally restricted, the referee can measure only operators $Q \in M_r$. Given ρ , the referee must guess ρ with a probability $\geq \eta \in (0, 1]$. Naturally, the referee minimizes the probability of guessing ρ when given $1/2^n$. Knowing the referee’s choice of Q (nontrivially, as many Q ’s can be optimal [74]), the agent tricks the referee by preparing a simulacrum $\tilde{\rho}$. The agent borrows $w \leq n$ uncomplex $|0\rangle$ ’s from an “uncomplexity bank” (e.g., someone else’s laboratory). The bank tacks on whichever $(n-w)$ -qubit state σ is handy, to raise the total number of qubits to n . The agent does not know σ ’s form but can tell the bank which qubits to prepare in $|0\rangle$. The agent transforms the n -qubit state with $\leq r$ gates. The referee must guess, with a probability $\geq 1 - \delta \in (0, 1]$, that the output is ρ .

Theorem 2 (Uncomplexity expenditure). *Let ρ denote an arbitrary n -qubit state. Let r and δ be as above, and assume that $\delta \geq 2r\epsilon$. For every $\eta \in (0, 1]$, and for every $(n-w)$ -qubit state σ , ρ can be imitated with $w = n - H_c^{r,\eta}(\rho)$ uncomplex $|0\rangle$ ’s.*

Appendix B contains the proof. We expect that $n - H_c^{r,\eta}(\rho)$ uncomplex $|0\rangle$ ’s are necessary.

Uncomplexity monotones.—Monotones are functions f that quantify a resource’s monotonic decline under allowed operations. If ρ denotes an arbitrary state and \mathcal{E} denotes an arbitrary allowed operation, $f(\rho) \geq f(\mathcal{E}(\rho))$. Different monotones quantify a state’s usefulness in different tasks. For example, consider extracting work by thermalizing an arbitrary state ρ (analogously to extracting work from an expanding gas) or performing work to

prepare ρ (analogously to compressing a gas). The extractable and required work are monotones in a thermodynamic resource theory [88]. They resemble free energies and can differ; there is no “one monotone to rule them all” [20].

We prove that two functions are fuzzy-operation monotones in certain regimes. The conditions reflect the notorious difficulty of proving that complexity measures grow monotonically under random dynamics [6, 10, 13, 73, 85, 86, 89–92]. We invoke a *brickwork circuit*, formed as follows. In layer 1, layer 3, etc., a gate transforms qubits 1 and 2, a gate transforms qubits 3 and 4, etc. In each even-indexed layer, a gate transforms qubits 2 and 3, a gate transforms qubits 4 and 5, etc. Periodic boundary conditions impose a gate on qubits n and 1 in each even-indexed layer. Define the *brickwork complexity* $C_{\text{bw}}(|\psi\rangle)$ as the least number R of gates in any brickwork circuit that prepares a pure state $|\psi\rangle$. The *brickwork uncomplexity* is $C_{\max} - C_{\text{bw}}(|\psi\rangle)$.

Theorem 3 (Monotonicity of brickwork uncomplexity: informal). *Let $|\psi\rangle$ denote an arbitrary n -qubit pure state. The brickwork uncomplexity $C_{\max} - C_{\text{bw}}(|\psi\rangle)$ cannot increase under any fuzzy brickwork circuit \tilde{U} of $\geq n$ layers, except in a measure-0 set of $|\psi\rangle$ -preparation-and- \tilde{U} -sampling experiments.*

We formalize and prove the theorem in App. C. The proof extends random-circuit results in [6] and leverages assumption (i) in Definition 1 (fuzziness extends in all directions of the 2-qubit-gate space).

The second monotone involves the complexity entropy, which $\leq n$ for every n -qubit state ρ . The *complexity negentropy* $n - H_c^{r,\eta}(\rho)$ quantifies how far ρ looks from maximally mixed under limited-complexity measurements. The complexity negentropy declines monotonically in two cases (App. D). In each, $\eta = 1$, enabling us to apply the algebraic-geometry toolkit of [6]. If $\eta \ll 1$, a fuzzy gate can decrease $H_c^{r,\eta}$: The error tolerance η can absorb sufficiently small fuzziness. We therefore conjecture the complexity negentropy’s monotonicity where some function $\eta_0(\epsilon)$ lower-bounds η .

Conjecture 1 (Monotonicity of the complexity negentropy). *The complexity negentropy $n - H_c^{r,\eta}(|\psi\rangle)$ cannot increase under fuzzy operations, for all $r \in \mathbb{Z}_{\geq 0}$, if $\eta \geq \eta_0(\epsilon)$, for some function $\eta_0(\epsilon)$.*

Conclusions.—We have confirmed Brown and Susskind’s conjecture [10] that a resource theory of uncomplexity can be defined. The resource theory’s allowed operations balance random evolutions, which model chaotic systems, with the agency in resource theories—the agent chooses operations to perform. Using the resource theory, we formalize uncomplexity expenditure and extraction. The tasks’ optimal efficiencies, we quantify with a complexity entropy that we introduce. Finally, we identify two fuzzy-operation monotones for certain regimes. This work introduces into quantum complexity the resource-theory toolbox

that has garnered successes across quantum information theory [20].

Our resource theory deviates superficially from two holographic conventions. We invoke the circuit complexity, instead of Nielsen’s geometric distance; correspondingly, gates act in discrete time steps, whereas Hamiltonians act continuously. Our model is motivated by (i) quantum information theory, where discrete gates form circuits; (ii) the closeness of circuit complexity to Nielsen’s complexity [85]; and (iii) random circuits’ modeling of chaotic evolutions. Reasons (ii) and (iii) suggest that our results might extend from fuzzy gates to perturbed continuous-time evolutions.

This work establishes several opportunities for future research. First, the complexity entropy evades the shortcomings described below Definition 2. This entropy can quantify the efficiencies of computationally restricted tasks, such as data compression with few gates, beyond uncomplexity extraction and expenditure. Computing the complexity entropy may be difficult typically, however. The complexity entropy’s properties, and applications to information-theoretic tasks and one-shot thermodynamics, will be explored elsewhere [74].

Second, the complexity entropy suggests an operational answer to a question undergoing active research—how to define mixed-state complexity [73, 93–98]: Complexity quantifies the difficulty of extracting uncomplex $|0\rangle$ qubits. More precisely, the complexity entropy can anchor a version of the strong complexity introduced in [73].

Third, proving Conjecture 1 would solidify the complexity negentropy’s interpretation as a resource quantifier. Also, a proof would strengthen the converse in Theorem 1 to arbitrarily many fuzzy gates: One would evaluate the complexity entropy on ρ and on the post-circuit state, then invoke the entropy’s monotonicity. Similarly,

Theorem 4 may generalize to a wider regime.

Fourth, the resource-theory framework suggests many questions, logged in [16]. For example, can allowed operations interconvert any two states asymptotically (if arbitrarily many copies are available)? Furthermore, we anticipate connections with other resource theories focused on the difficulty of implementing unitaries, e.g., the resource theory of magic [99–103].

Fifth, the resource theory can impact holography, many-body physics, and quantum computation. One might reframe black-hole paradoxes quantitatively in terms of resource extraction and expenditure. Also, though fuzzy gates involve little randomness, they can serve as proxies for highly random, chaotic Hamiltonian dynamics. Uncomplexity’s monotonicity under fuzzy circuits is also expected to relate to the *switchback effect* [8, 10], which determines how perturbations affect complexity’s evolution. The present work, providing a quantitative resource theory of uncomplexity as a technical tool, is hoped to galvanize further studies of space-time’s uncomplexity.

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Appendix A UNCOMPLEXITY-EXTRACTION PROOF

First, we prove a lemma used in the proofs of Theorems 1 and 2: Suppose that an arbitrary state ω undergoes a desired r -gate unitary U_r or a fuzzy approximation, \tilde{U}_r . The transformed states, $U_r\omega U_r^\dagger$ and $\tilde{U}_r\omega \tilde{U}_r^\dagger$, are $r\epsilon$ -close in trace distance. Then, we complete the proof of Theorem 1.

Lemma 1. *Let ω denote an arbitrary n -qubit state. Consider transforming ω by perfectly implemented gates V_1, V_2, \dots, V_r . Each V_j is defined on \mathbb{C}^{2n} but transforms just one qubit subspace nontrivially. The gates effect the unitary $U_r := V_r V_{r-1} \dots V_1$ and yield the state $U_r\omega U_r^\dagger$. Suppose that the gates implemented are ϵ -fuzzy. Analogously, denote the fuzzy gates by $\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_r$ and the effected unitary by $\tilde{U}_r := \tilde{V}_r \tilde{V}_{r-1} \dots \tilde{V}_1$. The fuzzy gates yield the state $\tilde{U}_r\omega \tilde{U}_r^\dagger$. The transformed states are $r\epsilon$ -close in trace distance:*

$$\mathcal{T}(U_r\omega U_r^\dagger, \tilde{U}_r\omega \tilde{U}_r^\dagger) \leq r\epsilon. \quad (\text{A1})$$

Proof. We prove the lemma in two steps. First, consider transforming an arbitrary n -qubit state τ with a perfectly implemented gate V_j or a fuzzy gate \tilde{V}_j . The fuzzy gates are ϵ -close to the perfect gates in operator norm, consistently with Definition 1:

$$\|\tilde{V}_j - V_j\|_\infty \leq \epsilon \quad \forall j = 1, 2, \dots, r. \quad (\text{A2})$$

Therefore, the transformed states are close:

$$\mathcal{T}\left(V_j\tau V_j^\dagger, \tilde{V}_j\tau\tilde{V}_j^\dagger\right) \leq \mathcal{T}\left(\tilde{V}_j\tau\tilde{V}_j^\dagger, V_j\tau\tilde{V}_j^\dagger\right) + \mathcal{T}\left(V_j\tau\tilde{V}_j^\dagger, V_j\tau V_j^\dagger\right) \quad (\text{A3})$$

$$= \frac{1}{2} \left\| \left(\tilde{V}_j - V_j \right) \tau \tilde{V}_j^\dagger \right\|_1 + \frac{1}{2} \left\| V_j \tau \left(\tilde{V}_j - V_j \right)^\dagger \right\|_1 \quad (\text{A4})$$

$$= \frac{1}{2} \left\| \left(\tilde{V}_j - V_j \right) \tau \right\|_1 + \frac{1}{2} \left\| \tau \left(\tilde{V}_j - V_j \right)^\dagger \right\|_1 \quad (\text{A5})$$

$$= \|(\tilde{V}_j - V_j)\tau\|_1 \quad (\text{A6})$$

$$\leq \|\tilde{V}_j - V_j\|_\infty \|\tau\|_1 \quad (\text{A7})$$

$$\leq \epsilon. \quad (\text{A8})$$

Inequality (A3) follows from the triangle inequality. Equation (A5) follows from the trace distance's unitary invariance. Equation (A6) follows from the property $\|A\|_1 = \|A^\dagger\|_1$ of all linear operators A . Inequality (A7) follows from Hölder's inequality. Inequality (A8) follows from (A2) and from $\|\tau\|_1 = 1$.

Second, we prove Ineq. (A1) inductively. The bound (A1) holds when $r = 1$:

$$\mathcal{T}\left(U_r\omega U_r^\dagger, \tilde{U}_r\omega\tilde{U}_r^\dagger\right) = \mathcal{T}\left(V_1\omega V_1^\dagger, \tilde{V}_1\omega\tilde{V}_1^\dagger\right) \leq \epsilon. \quad (\text{A9})$$

Suppose that Ineq. (A3) holds for all $r' = 1, 2, \dots, r-1$. Define $U_{r-1} := V_{r-1}V_{r-2}\dots V_1$ and $\tilde{U}_{r-1} := \tilde{V}_{r-1}\tilde{V}_{r-2}\dots \tilde{V}_1$. We can use these unitaries to bound the original trace distance:

$$\begin{aligned} \mathcal{T}\left(U_r\omega U_r^\dagger, \tilde{U}_r\omega\tilde{U}_r^\dagger\right) &\leq \mathcal{T}\left(V_r U_{r-1}\omega U_{r-1}^\dagger V_r^\dagger, V_r \tilde{U}_{r-1}\omega\tilde{U}_{r-1}^\dagger V_r^\dagger\right) \\ &\quad + \mathcal{T}\left(V_r \tilde{U}_{r-1}\omega\tilde{U}_{r-1}^\dagger V_r^\dagger, \tilde{V}_r \tilde{U}_{r-1}\omega\tilde{U}_{r-1}^\dagger \tilde{V}_r^\dagger\right) \end{aligned} \quad (\text{A10})$$

$$\leq (r-1)\epsilon + \epsilon \quad (\text{A11})$$

$$= r\epsilon. \quad (\text{A12})$$

Inequality (A10) follows from the triangle inequality. Inequality (A11) follows from (i) the trace distance's unitary invariance, (ii) the inductive hypothesis, and (iii) the application of Ineq. (A8) to $\tau = \tilde{U}_{r-1}\omega\tilde{U}_{r-1}^\dagger$. \square

Having proved a lemma used in Theorems 1 and 2, we complete the proof of Theorem 1.

Proof. First, we prove that extracting $w = n - H_c^{r,\eta}(\rho)$ uncomplex $|0\rangle$'s from ρ is achievable. Then, we show that this number is optimal: No protocol can extract more $|0\rangle$'s.

Achievability: Let ρ denote any n -qubit state, $\eta \in (0, 1]$, and $r \in \mathbb{Z}_{\geq 0}$. Consider any Q that achieves the minimization in Eq. (2). Q projects $n - H_c^{r,\eta}(\rho)$ qubits onto $|0\rangle$. Without loss of generality, we index those qubits as $1, 2, \dots, n - H_c^{r,\eta}(\rho)$. Denote that set of qubits by W ; and the rest of the qubits, by \bar{W} . Denote by U_r the ($\leq r$)-qubit unitary used to implement Q : $Q = U_r^\dagger (|0^{n-H_c^{r,\eta}(\rho)}\rangle\langle 0^{n-H_c^{r,\eta}(\rho)}| \otimes \mathbb{1}^{\otimes H_c^{r,\eta}(\rho)}) U_r$.

By the constraint in the definition (2),

$$\eta \leq \text{Tr}(Q\rho) = \text{Tr}\left(U_r^\dagger \left[|0^{n-H_c^{r,\eta}(\rho)}\rangle\langle 0^{n-H_c^{r,\eta}(\rho)}| \otimes \mathbb{1}^{\otimes H_c^{r,\eta}(\rho)} \right] U_r \rho\right) \quad (\text{A13})$$

$$= \text{Tr}_{\bar{W}}\left(\text{Tr}_W(U_r \rho U_r^\dagger) |0^{n-H_c^{r,\eta}(\rho)}\rangle\langle 0^{n-H_c^{r,\eta}(\rho)}| \right). \quad (\text{A14})$$

The trace's cyclicity implies the final equality. Equation (A14) implies that $\text{Tr}_W(U_r \rho U_r^\dagger)$ has a fidelity $\geq \eta$ to $|0^{n-H_c^{r,\eta}(\rho)}\rangle\langle 0^{n-H_c^{r,\eta}(\rho)}|$. By the relationship between the fidelity and the trace distance [104, Theorem 9.3.1],

$$\mathcal{T}(\text{Tr}_W(U_r \rho U_r^\dagger), |0^{n-H_c^{r,\eta}(\rho)}\rangle\langle 0^{n-H_c^{r,\eta}(\rho)}|) \leq \sqrt{1-\eta}. \quad (\text{A15})$$

Therefore, if the gates are implemented perfectly, the protocol extracts $w = n - H_c^{r,\eta}(\rho)$ uncomplex $|0\rangle$'s with accuracy $\leq \sqrt{1-\eta}$.

Now, suppose that the gates are ϵ -fuzzy. Attempting to implement U_r , the agent actually implements an approximation \tilde{U}_r . By Lemma 1, $\mathcal{T}(U_r \rho U_r^\dagger, \tilde{U}_r \rho \tilde{U}_r^\dagger) \leq r\epsilon$. The trace distance is contractive under all completely positive,

trace-preserving maps, including the partial trace. Therefore, $\mathcal{T}(\text{Tr}_W(U_r\rho U_r^\dagger), \text{Tr}_W(\tilde{U}_r\rho\tilde{U}_r^\dagger)) \leq r\epsilon$. We combine this inequality with Ineq. (A15), using the triangle inequality:

$$\mathcal{T}(\text{Tr}_W(\tilde{U}_r\rho\tilde{U}_r^\dagger), |0^{n-H_c^{r,\eta}(\rho)}\rangle\langle 0^{n-H_c^{r,\eta}(\rho)}|) \leq \mathcal{T}(\text{Tr}_W(\tilde{U}_r\rho\tilde{U}_r^\dagger), \text{Tr}_W(U_r\rho U_r^\dagger)) \quad (\text{A16})$$

$$+ \mathcal{T}(\text{Tr}_W(U_r\rho U_r^\dagger), |0^{n-H_c^{r,\eta}(\rho)}\rangle\langle 0^{n-H_c^{r,\eta}(\rho)}|) \\ \leq \sqrt{1-\eta} + r\epsilon. \quad (\text{A17})$$

Therefore, for $\delta \geq \sqrt{1-\eta} + r\epsilon$, or $\eta \geq 1 - (\delta - r\epsilon)^2$, one can extract $w = n - H_c^{r,\eta}(\rho)$ uncomplex $|0\rangle$'s, with accuracy $\geq \delta$, using ϵ -fuzzy operations.

Optimality: Again, denote by W the set of not-discarded qubits, and index them as the first qubits. Denote by \bar{W} the set of discarded qubits. By the constraints on the extraction protocol, the final state must be δ -close to the $|0^w\rangle\langle 0^w|$: If \tilde{U}_r denotes the circuit performed with $\leq r$ fuzzy gates,

$$\mathcal{T}(\text{Tr}_{\bar{W}}(\tilde{U}_r\rho\tilde{U}_r^\dagger), |0^w\rangle\langle 0^w|) \leq \delta. \quad (\text{A18})$$

We can bound this expression using the following quantum-information result: Let σ and γ denote quantum states defined on the same Hilbert space, and let Λ denote an operator such that $0 \leq \Lambda \leq \mathbb{1}$. According to Corollary 9.1.1 of Ref. [104],

$$\mathcal{T}(\sigma, \gamma) \geq \text{Tr}(\Lambda\gamma) - \text{Tr}(\Lambda\sigma). \quad (\text{A19})$$

Let $\sigma = \text{Tr}_W(\tilde{U}_r\rho\tilde{U}_r^\dagger)$, and let $\gamma = \Lambda = |0^w\rangle\langle 0^w|$. We substitute into Ineq. (A19), combine the result with Ineq. (A18), and rearrange terms. The result is

$$\text{Tr}(|0^w\rangle\langle 0^w| \text{Tr}_{\bar{W}}(\tilde{U}_r\rho\tilde{U}_r^\dagger)) \geq 1 - \delta. \quad (\text{A20})$$

Let us rewrite the trace's argument such that each factor is defined on the n -qubit Hilbert space. Padding the outer product with identity operators at the discarded sites yields

$$|0^w\rangle\langle 0^w| \otimes \mathbb{1}^{\otimes(n-w)} =: Q_0. \quad (\text{A21})$$

Therefore, by Eq. (A20), $\text{Tr}(Q_0[\tilde{U}_r\rho\tilde{U}_r^\dagger]) \geq 1 - \delta$. Let us cycle the \tilde{U}_r^\dagger leftward. Packaging up $\tilde{U}_r^\dagger Q_0 \tilde{U}_r =: \bar{Q}$ implies $\text{Tr}(\bar{Q}\rho) \geq 1 - \delta$. By the foregoing inequality, and by the number of the fuzzy gates that constitute U_r , \bar{Q} is in M_r . By the minimum in Eq. (3),

$$H_c^{r,1-\delta}(\rho) \leq \log_2(\text{Tr}(\bar{Q})) = \log_2(2^{n-w}) = n - w. \quad (\text{A22})$$

The penultimate equality follows from Eq. (A21). \square

Appendix B UNCOMPLEXITY-EXPENDITURE PROOF

This appendix contains the proof of Theorem 2. We must upper-bound the cost of simulating a state ρ δ -approximately. First, we prove Theorem 2 in the absence of fuzziness (Lemma 2). Then, we use Lemma 1 to extend the proof to fuzzy gates.

Lemma 2. *Let ρ denote an arbitrary n -qubit state. Let r and δ be as described above Theorem 2, but let all gates be implemented perfectly ($\epsilon = 0$). For every $\delta \in (0, 1]$, every $\eta \in (0, 1]$, every $(n - w)$ -qubit state σ , ρ can be imitated with $w = n - H_c^{r,\eta}(\rho)$ uncomplex $|0\rangle$'s.*

Proof. We index the qubits such that the referee's measurement operator has the form

$$Q_{\text{ref}} = U_r^\dagger (|0^{w'}\rangle\langle 0^{w'}| \otimes \mathbb{1}^{\otimes(n-w')}) U_r, \quad (\text{B1})$$

for some $w' \in \{1, 2, \dots, n\}$ and some unitary U_r implementable with $\leq r$ gates. By the constraints on the referee, $\text{Tr}(Q_{\text{ref}}\rho) \geq \eta$. Hence Q_{ref} satisfies the constraint in the definition 2 of $H_c^{r,\eta}(\rho)$.

Let the agent request $w = w'$ uncomplex $|0\rangle$'s. The agent can perform the inverse U_r^\dagger of the referee's unitary. The simulacrum acquires the form

$$\bar{\rho} = U_r^\dagger (|0^w\rangle\langle 0^w| \otimes \sigma) U_r. \quad (\text{B2})$$

If the referee receives $\bar{\rho}$, their probability of guessing ρ is $\text{Tr}(Q_{\text{ref}}\bar{\rho}) = 1 \geq 1 - \delta$. Hence $\bar{\rho}$ satisfies the constraint on the agent.

We can derive two expressions for $\log_2(\text{Tr}(Q_{\text{ref}}))$. First, by Eq. (B1) and $w' = w$, $\log_2(\text{Tr}(Q_{\text{ref}})) = n - w$. Second, Q_{ref} was chosen to minimize $\text{Tr}(Q_{\text{ref}} \mathbb{1}/2^n)$. Therefore, Q_{ref} achieves the minimum in the definition (2). Therefore, $\log_2(\text{Tr}(Q_{\text{ref}})) = H_c^{r,\eta}(\rho)$. Equating the two expressions for the log, and solving for w , yields $w = n - H_c^{r,\eta}(\rho)$. \square

We have proved a fuzziness-free variation on Theorem 2. We extend that proof to a proof of the theorem itself, assuming that all gates applied are ϵ -fuzzy.

Proof. Recall the assumption that $\delta \geq 2r\epsilon$. Due to gate fuzziness, the referee implements the fuzzy operation \tilde{U}_r^{ref} , instead of U_r , and effects the POVM \tilde{Q} , instead of \bar{Q} . Likewise, the agent implements the fuzzy operation \tilde{U}_r^{agt} , instead of U_r , and constructs the state $\tilde{\rho}$, instead of $\bar{\rho}$. The referee identifies $\tilde{\rho}$ as ρ with probability at least $1 - \delta$, since

$$\text{Tr}(\tilde{Q}\tilde{\rho}) = \text{Tr}\left(\left\{\tilde{U}_r^{\text{ref}\dagger} [|0^w\rangle\langle 0^w| \otimes \mathbb{1}^{\otimes(n-w)}] \tilde{U}_r^{\text{ref}}\right\} \left\{\tilde{U}_r^{\text{agt}\dagger} [|0^w\rangle\langle 0^w| \otimes \sigma] \tilde{U}_r^{\text{agt}}\right\}\right) \quad (\text{B3})$$

$$= \text{Tr}\left([|0^w\rangle\langle 0^w| \otimes \mathbb{1}^{\otimes(n-w)}] \tilde{U}_r^{\text{ref}} \tilde{U}_r^{\text{agt}\dagger} [|0^w\rangle\langle 0^w| \otimes \sigma] \tilde{U}_r^{\text{agt}} \tilde{U}_r^{\text{ref}\dagger}\right) \quad (\text{B4})$$

$$\geq \text{Tr}\left([|0^w\rangle\langle 0^w| \otimes \mathbb{1}^{\otimes(n-w)}] [|0^w\rangle\langle 0^w| \otimes \sigma]\right) \\ - \mathcal{T}\left(|0^w\rangle\langle 0^w| \otimes \sigma, \tilde{U}_r^{\text{ref}} \tilde{U}_r^{\text{agt}\dagger} [|0^w\rangle\langle 0^w| \otimes \sigma] \tilde{U}_r^{\text{agt}} \tilde{U}_r^{\text{ref}\dagger}\right) \quad (\text{B5})$$

$$= 1 - \mathcal{T}\left(\tilde{U}_r^{\text{ref}\dagger} [|0^w\rangle\langle 0^w| \otimes \sigma] \tilde{U}_r^{\text{ref}}, \tilde{U}_r^{\text{agt}\dagger} [|0^w\rangle\langle 0^w| \otimes \sigma] \tilde{U}_r^{\text{agt}}\right) \quad (\text{B6})$$

$$\geq 1 - \mathcal{T}\left(\tilde{U}_r^{\text{ref}\dagger} [|0^w\rangle\langle 0^w| \otimes \sigma] \tilde{U}_r^{\text{ref}}, U_r^\dagger [|0^w\rangle\langle 0^w| \otimes \sigma] U_r\right) \\ - \mathcal{T}\left(U_r^\dagger [|0^w\rangle\langle 0^w| \otimes \sigma] U_r, \tilde{U}_r^{\text{agt}\dagger} [|0^w\rangle\langle 0^w| \otimes \sigma] \tilde{U}_r^{\text{agt}}\right) \quad (\text{B7})$$

$$\geq 1 - 2r\epsilon \quad (\text{B8})$$

$$\geq 1 - \delta. \quad (\text{B9})$$

Inequality (B5) follows by the application of Ineq. (A19) to $\sigma = \tilde{U}_r^{\text{ref}} \tilde{U}_r^{\text{agt}\dagger} [|0^w\rangle\langle 0^w| \otimes \sigma] \tilde{U}_r^{\text{agt}} \tilde{U}_r^{\text{ref}\dagger}$, $\gamma = |0^w\rangle\langle 0^w| \otimes \sigma$, and $\Lambda = |0^w\rangle\langle 0^w| \otimes \mathbb{1}^{\otimes(n-w)}$. Equation (B6) follows the trace distance's unitary invariance. Inequality (B7) follows from the triangle inequality. Inequality (B8) follows from Lemma 1. Therefore, the agent can imitate ρ with probability $\geq 1 - \delta$ using (as shown in Lemma 2) $w = n - H_c^{r,\delta}(\rho)$ uncomplex $|0\rangle$'s. \square

Appendix C PROOF OF THE BRICKWORK COMPLEXITY'S MONOTONICITY

This appendix contains the proof of Theorem 4. We extend results in Ref. [6] from Haar-random gates to fuzzy gates, then to monotonicity statements.

Several pieces of background are necessary. We call an arrangement of gates an *architecture*. Slotted particular gates into an architecture produces a circuit. A circuit may contain a *light cone*, a block of gates the contains one qubit that links to each other qubit via a path, perhaps unique to the latter qubit, formed from gates (Fig. 2 of [6]).

An *accessible dimension* is introduced in Ref. [6]: Consider choosing two-qubit gates whose fuzzy approximations are slotted into any architecture A . The slotting-in forms a circuit that implements a unitary. All the unitaries so implementable form a set $\mathcal{U}(A)$. The number of degrees of freedom needed to specify $\mathcal{U}(A)$ is the *accessible dimension* $\dim(\mathcal{U}(A)) \leq 4^n$ [6]. Consider applying to $|0^n\rangle$ each unitary in $\mathcal{U}(A)$. The resulting states form a set that we denote by $\mathcal{U}_{\text{state}}(A)$.

A fuzzy gate, recall, is drawn from the group $SU(4)$ of two-qubit gates. Denote by $d\tilde{U}$ the Haar measure on $SU(4)$. By definition, fuzzy gates are drawn according to probability distributions of the form $p_{U,\epsilon}(\tilde{U}) d\tilde{U}$, for a density function $p_{U,\epsilon}(\tilde{U})$ in L^1 . Therefore, the fuzzy-gate distribution is *absolutely continuous* with respect to the Haar measure: Consider any set of gates that has measure 0 with respect to the Haar measure on $SU(4)^{\times R}$. The set has measure 0 also for the fuzzy-gate distribution.

By an n -qubit Pauli string, we mean, an n -fold tensor product of (i) one-qubit Pauli operators and (ii) one-qubit identity operators $\mathbb{1}$. We prove the following technical version of Theorem 3.

Theorem 4 (Monotonicity of exact uncomplexity: formal). *Consider the following protocol: Choose any two-qubit gates U_1, U_2, \dots, U_R , for some $R \in \mathbb{Z}_+$. Let $\epsilon' > 0$, and draw $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_R$ near U_1, U_2, \dots, U_R according to an ϵ' -fuzzy distribution over $SU(4)^{\times R}$. Slot the resulting gates into any brickwork architecture. Apply the resulting circuit to*

$|0^n\rangle$. This protocol prepares a state whose brickwork uncomplexity is $\geq \mathcal{C}_{\max} - R$. For every integer $R \in (0, \Omega(4^n))$, the following holds: Let $V_1, V_2, \dots, V_{n(n-1)}$ denote any two-qubit gates in an n -layer brickwork architecture. Let $\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_{n(n-1)}$ denote corresponding ϵ -fuzzy gates, with $\epsilon > 0$. With probability 1, the brickwork uncomplexity decreases:

$$\mathcal{C}_{\max} - \mathcal{C}_{\text{bw}} \left(\left[\tilde{V}_{n(n-1)} \tilde{V}_{n(n-1)-1} \dots \tilde{V}_1 \right] \left[\tilde{U}_R \tilde{U}_{R-1} \dots \tilde{U}_1 \right] |0^n\rangle \right) < \mathcal{C}_{\max} - R. \quad (\text{C1})$$

ϵ and ϵ' can be chosen to be arbitrarily small without affecting the theorem. To connect the formal statement above with the main text's informal statement (Theorem 3), we proceed as follows. We choose for the unitaries U_1, U_2, \dots, U_R to form an optimal brickwork preparation circuit for $|\psi\rangle$. We choose for ϵ' to be much smaller than other parameters in the problem. The latter choice ensures that the state transformed by $V_1, V_2, \dots, V_{n(n-1)}$ is arbitrarily close to $|\psi\rangle$.

We need the following lemma, proven as Lemma 1 in Ref. [6]:

Lemma 3. *Let A denote any architecture. The states preparable with architecture- A circuits form the set $\mathcal{U}_{\text{state}}(A)$. Let $M \subset \mathcal{U}_{\text{state}}(A)$ denote any (semialgebraic) subset for which $\dim(M) < \dim(\mathcal{U}_{\text{state}}(A))$. Consider drawing an architecture- A circuit uniformly randomly. The circuit effects a unitary in M with probability 0.*

Proof of Theorem 4. To prove that the brickwork uncomplexity decreases monotonically, we prove that the brickwork complexity increases monotonically. Denote by $A_{\text{bw},T}$ the T -layer brickwork architecture, which contains $T(n-1) = R$ gates total. To prove Ineq. (C1), we must prove only that

$$\dim(\mathcal{U}_{\text{state}}(A_{\text{bw},T+n})) > \dim(\mathcal{U}_{\text{state}}(A_{\text{bw},T})). \quad (\text{C2})$$

The reason is, Ineq. (C2) and Lemma 3 imply the following: Consider randomly drawing a unitary effected by a brickwork architecture $A_{\text{bw},T+n}$. The unitary has zero probability of being implementable with a brickwork architecture whose $R = T(n-1)$.

Let us prove Ineq. (C2). Consider contracting the gates in $A_{\text{bw},T}$, then applying the resulting unitary to $|0^n\rangle$. We map a set of R two-qubit gates to a point on the unit sphere. More generally, contraction forms a smooth map $f : \text{SU}(4)^{\times R} \rightarrow S^{2 \times 2^n - 1}$. The map's greatest possible rank equals $\dim(\mathcal{U}_{\text{state}}(A))$, we show via semialgebraic geometry in Ref. [6]. The map's rank also—by definition—equals the rank of the map's Jacobian. Therefore, to prove Ineq. (C2), we must identify one circuit—one point $x = (U_1, U_2, \dots, U_R, U_{R+1}, \dots, U_{R+n(n-1)}) \in \text{SU}(4)^{\times[R+n(n-1)]}$ —for which the map's Jacobian has a rank $> \dim(\mathcal{U}_{\text{state}}(A_{\text{bw},T}))$.

We construct that circuit as follows. Let P_j denote a two-local Pauli operator that acts on the same qubits as U_j . The Jacobian's image is spanned by $\{(U'_R U'_{R-1} \dots U'_{j+1}) P_j (U'_j U'_{j-1} \dots U'_1)\}_{j,P}$ [6]. Denote by $x_{\max} \in \text{SU}(4)^{\times R}$ a point at which the A contraction map achieves its maximal rank, r_{\max} . We can construct a point $x_{\text{ext}} = (U'_1, U'_2, \dots, U'_R, V'_1, V'_2, \dots, V'_{n(n-1)})$ at which the contraction map's rank exceeds the rank at r_{\max} . (The V'_j 's are gates chosen to prove a variation on Theorem 4. In the variation, Haar-random gates replace the fuzzy gates \tilde{V}_j . By absolute continuity, the theorem follows for fuzzy gates.)

There exist Hermitian operators H_j such that the following is true: The map has a Jacobian whose image, at x_{\max} , is spanned by $\{H_j U_R U_{R-1} \dots U_1 |0^n\rangle\}_{j=1,2,\dots,r_{\max}} \equiv \{|v_j\rangle\}$. These vectors' span excludes some of the states formed by applying an n -qubit Pauli string to $U_R U_{R-1} \dots U_1 |0^n\rangle$.¹ The gates $V'_1, V'_2, \dots, V'_{n(n-1)}$ transform some excluded Pauli string P into Z_ℓ , for a to-be-specified qubit ℓ : $(V'_{n(n-1)} V'_{n(n-1)-1} \dots V'_1) P (V'_1{}^\dagger V'_2{}^\dagger \dots V'_{n(n-1)}{}^\dagger) = Z_\ell$. The foregoing claim is true, Ref. [6] shows, under two necessary conditions: (i) The gates $V'_1, V'_2, \dots, V'_{n(n-1)}$ are in an architecture that contains a light cone. (ii) ℓ indexes the light cone's final qubit, which connects to all the other light-cone qubits via paths formed from gates. Both conditions are satisfied by an $n(n-1)$ -gate brickwork circuit and $\ell = 1$. All the states $(V'_{n(n-1)} V'_{n(n-1)-1} \dots V'_1) H_j (V'_1{}^\dagger V'_2{}^\dagger \dots V'_{n(n-1)}{}^\dagger) (U'_R U'_{R-1} \dots U'_1) |0^n\rangle$ and $Z_\ell (U'_R U'_{R-1} \dots U'_1) |0^n\rangle$ (i) are in the Jacobian's image and (ii) are linearly independent by assumption. Therefore, the contraction map's Jacobian has a rank $\dim(\mathcal{U}_{\text{state}}(A_{\text{bw},T}))$.

Therefore, Ineq. (C2) holds for circuits extended with Haar-random gates V'_j . The fuzzy-gate probability distribution is absolutely continuous with respect to the Haar measure, as explained in the beginning of this appendix. Therefore, Theorem 4 is true if fuzzy gates replace the Haar-random gates. \square

¹ We can prove this claim by contradiction: Assume that, for all Pauli operators P , $P U_R U_{R-1} \dots U_1 |0^n\rangle \in \text{span}\{|v_j\rangle\}$. The Pauli operators form an orthonormal basis for the Hermitian operators defined on the same Hilbert space. Therefore, for every state $|\psi\rangle$ in the space, we can build the Hermitian operator $A' = |\psi\rangle\langle 0^n| (U'_1 U'_2 \dots U'_r) + \text{h.c.}$ By the assumption we mean to

contradict, $A' (U'_R U'_{R-1} \dots U'_1) |0^n\rangle = |\psi\rangle \in \text{span}\{|v_j\rangle\}$.

However, we can derive a contradiction to the foregoing equation. According to the text above, the Jacobian's image has a submaximal rank. Therefore, $\text{span}\{|v_j\rangle\}$ is not the entire space. Therefore, we can choose for $|\psi\rangle$ to be orthogonal to the $|v_j\rangle$. However, we have already proved that $|\psi\rangle \in \text{span}\{|v_j\rangle\}$. We have proven a contradiction, so our first premise is false.

Appendix D MONOTONICITY OF THE COMPLEXITY ENTROPY IN TWO CASES

This section supports our conjecture that the complexity negentropy declines monotonically under fuzzy operations (Conjecture 1). We show that the complexity entropy grows monotonically in two cases. In both, the error-intolerance parameter $\eta = 1$. The reason is, our techniques derive from a proof of the linear growth, under random circuits, of the *exact complexity*, the least number of two-qubit gates required to prepare an n -qubit state $|\psi\rangle$ from $|0^n\rangle$ exactly [6]. Define as the *approximate complexity* the least number of two-qubit gates required to prepare $|\psi\rangle$ approximately. We would have to prove the approximate complexity's linear growth, to prove that $H_c^{r,\eta}$ increases monotonically under fuzzy operations when $\eta < 1$. The approximate proof would require more than the dimension counting used here; we would need insights into the geometry of the set of unitaries implemented by local quantum circuits.

Two definitions underlie our proof: (i) Define an *architecture* as the layout of gates in a quantum circuit. (ii) Define as $\mathcal{E}_{k,r}$ the ensemble of n -qubit states formed as follows: Pick $k = 0, 1, \dots, n$ qubits uniformly randomly; and pick a k -qubit state vector $|\phi\rangle$ Haar-randomly. Prepare those qubits in $|\phi\rangle$. Pad $|\phi\rangle$ with $|0\rangle$'s, to produce $|\phi\rangle|0^{n-k}\rangle$. Perform a circuit, with a random ($\leq r$)-gate architecture, of any fuzzy gates.²

Lemma 4. *Let the number of chosen qubits be $k > \log_2(15r)$. Consider drawing a state from $\mathcal{E}_{k,r}$ uniformly randomly, then performing an arbitrary fuzzy gate. The complexity negentropy $n - H_c^{r,0}$ does not increase, with probability 1.*

Proof. Consider the “uncomplex” POVM elements $Q \in M_r$, for which $\log_2 \text{Tr}(Q) \leq k$. Consider the states $|\psi\rangle$ for which, for some such Q , $\text{Tr}(Q|\psi\rangle\langle\psi|) = 1$. All the $Q \in M_r$ are mutually orthogonal projectors. Therefore, the aforementioned $|\psi\rangle$'s form the set

$$\mathcal{U}_{r,k} := \bigcup_{Q \in M_r} \text{Im}(Q) = \bigcup_{Q \in M_r} \bigcup_{\pi \in S_n} \{U_r U_{r-1} \dots U_1 \pi |\phi\rangle|0^{n-k}\rangle : U_j \text{ is two-local } \forall j, |\phi\rangle \in (\mathbb{C}^2)^{\otimes k}\}. \quad (\text{D1})$$

The projector Q has an image $\text{Im}(Q)$, and S_n denotes the group of the permutations of n objects. Since $|0^k\rangle$ is a state $|\phi\rangle \in \mathbb{C}^k$, $\mathcal{U}_{r,0} \subseteq \mathcal{U}_{r,1} \subseteq \dots \subseteq \mathcal{U}_{r,k}$.

To characterize the \mathcal{U} 's further, we denote by S^D the sphere in \mathbb{R}^{D+1} .³ Also, we identify \mathbb{C}^{2^k} with $\mathbb{R}^{2 \times 2^k}$. The $\mathcal{U}_{r,k}$ are the images of the polynomial contraction maps $S^{2 \times 2^k-1} \times \text{SU}(4)^{\times R} \rightarrow (\mathbb{C}^2)^{\otimes k}$. Therefore, by the Tarski-Seidenberg principle [105], the $\mathcal{U}_{r,k}$ are semialgebraic sets. (A semialgebraic set consists of the solutions to a finite set of polynomial equations and inequalities over the real numbers.) Every semialgebraic set decomposes as a union of smooth manifolds [105]. The greatest manifold dimension is the semialgebraic set's dimension [105].

Therefore, we can bound $\dim(\mathcal{U}_{r,k})$ as follows. Since $\{|\phi\rangle|0^{n-k}\rangle\} \subseteq \mathcal{U}_{r,k}$, $\dim(\mathcal{U}_{r,k}) \geq \dim(S^{2 \times 2^k-1}) = 2 \times 2^k - 1$. According to Lemma 4, $k > \log_2(15r)$. Therefore, $2 \times 2^k - 1 > 2 \times 2^{k-1} - 1 + 15r$. By parameter counting, $\dim(\mathcal{U}_{r,k-1}) \leq 2 \times 2^{k-1} - 1 + 15r$. Therefore, $\dim(\mathcal{U}_{r,k-1}) < \dim(\mathcal{U}_{r,k})$. Therefore, by Ref. [6, Lemma 1], $\mathcal{U}_{r,k-1}$ forms a measure-0 set in $\mathcal{U}_{r,k}$.

We apply the above conclusion as follows. Consider drawing a state uniformly randomly from the ensemble $\mathcal{E}_{k,r}$. If the state has a complexity entropy $H_c^{r,0} \leq k - 1$, the state is in $\mathcal{U}_{r,k-1}$, which forms a measure-0 set in $\mathcal{U}_{r,k}$, we just concluded. Therefore, the drawn state satisfies $H_c^{r,0} \leq k - 1$ with probability 0. Applying a unitary to the $\mathcal{U}_{r,k}$ elements constitutes a diffeomorphism. Therefore, the unitary does not change the set's dimension. Fuzzy operations are unitaries, so applying a fuzzy operation cannot decrease $H_c^{r,0}$. \square

Having proved that the complexity negentropy decreases monotonically in one case, we proceed to the second case. Denote by $\mathcal{E}_{k,A}$ the ensemble defined as $\mathcal{E}_{k,r}$, except that the r -gate fuzzy circuit has the architecture A .

Lemma 5. *For every $k < n$ and $r = 0, 1, \dots, 2 \times \lfloor (2^n - 1 - 2^k)/15 \rfloor$, there exists an architecture A for which the following holds: Let A' denote any architecture that contains a light cone. Consider drawing (i) a state from $\mathcal{E}_{k,A}$ and (ii) an architecture- A' circuit. Consider following A with the light-cone-containing A' , and call the extended architecture A_{ext} . Running the circuit on the state decreases the complexity negentropy $n - H_c^{r,0}$, with probability 1 over the ensemble $\mathcal{E}_{k,A_{\text{ext}}}$.*

² We draw the random architecture, or directed graph, as follows:

Sequentially draw R uniformly random pairs of qubits (j, k) , with $j \neq k$. For each pair, add to the graph a vertex between edges j

and k .

³ Our proof could be cast equivalently in terms of complex projective spaces, which are prevalent in the holographic literature [10].

Proof. Define, similarly to Eq. (D1), the set

$$\mathcal{U}_{k,A} := \bigcup_{Q \in M_r} \{U_r U_{r-1} \dots U_1 |\phi\rangle |0^{n-k}\rangle : \text{circuit has architecture } A, |\phi\rangle \in (\mathbb{C}^2)^{\otimes k}\}. \quad (\text{D2})$$

We proceed similarly to the proof of Theorem 4: Let the architecture A be such that $\dim(\mathcal{U}_{k,A})$ has the greatest value achievable with any ($\leq r$)-gate architecture. We must prove only that $\dim(\mathcal{U}_{k,A}) < \dim(\mathcal{U}_{k,A_{\text{ext}}})$: If this inequality holds, then, by Lemma 3, randomly drawing a state from $\mathcal{E}_{k,A_{\text{ext}}}$ has 0 probability of being in $\mathcal{U}_{k,\tilde{A}}$, for every architecture \tilde{A} with $\leq r$ gates.

The dimension $\dim(\mathcal{U}_{k,A})$ equals the dimension of a contraction map's Jacobian. The Jacobian's image is spanned by a set of vectors $|v_j\rangle \in \mathbb{R}^{2 \times 2^n - 1}$. We can choose for the vectors to have the form $|v_j\rangle = A_j U_1 U_2 \dots U_r |0^{n-k}\rangle |\phi\rangle$, for Hermitian operators A_j . There is a Pauli operator P such that $|v'\rangle = P U_1 U_2 \dots U_r |0^{n-k}\rangle |\phi\rangle \notin \text{span}\{|v_j\rangle\}$, if the rank of the Jacobian's image does not attain the greatest possible value, $2 \times 2^n - 1$.⁴ The rank's submaximality is guaranteed by the assumptions $k < n$ and $r = 0, 1, \dots, 2 \times \lfloor (2^n - 1 - 2^k)/15 \rfloor$. We can follow A with an architecture- A' , depth- R' circuit. Applying the procedure of Ref. [6] to the Pauli operator P , we find a higher-rank point for A_{ext} : $\dim(\mathcal{U}_{k,A}) < \dim(\mathcal{U}_{k,A_{\text{ext}}})$. \square

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⁴ This claim follows as in the proof of Theorem 4.

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