How to build Hamiltonians that transport noncommuting charges in quantum thermodynamics

Nicole Yunger Halpern[1, 2, 3, 4, 5, 6, *] and Shayan-Shawn Majidy[7, †]

[1] ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138, USA
[2] Department of Physics, Harvard University, Cambridge, MA 02138, USA
[3] Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
[4] Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
[5] Joint Center for Quantum Information and Computer Science, NIST and University of Maryland, College Park, MD 20742, USA
[6] Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA
[7] Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

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Noncommuting conserved quantities have recently launched a subfield of quantum thermodynamics. In conventional thermodynamics, a system of interest and a bath exchange quantities—energy, particles, electric charge, etc.—that are globally conserved and are represented by Hermitian operators. These operators were implicitly assumed to commute with each other, until a few years ago. Freeing the operators to fail to commute has enabled many theoretical discoveries—about reference frames, entropy production, resource-theory models, etc. Little work has bridged these results from abstract theory to experimental reality. This paper provides a methodology for building this bridge systematically: We present an algorithm for constructing Hamiltonians that conserve noncommuting quantities globally while transporting the quantities locally. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. Special cases of our construction have appeared in quantum chromodynamics (QCD). Our Hamiltonians may be realized physically with superconducting qudits, with ultracold atoms, with trapped ions, and in QCD.

Noncommuting conserved quantities have been enjoying a heyday in quantum thermodynamics [1–23]. They introduce a nonclassical twist into a common problem: Throughout statistical mechanics, a small system S is often assumed to exchange quantities with a bath B. The exchanged quantities are conserved globally, so we call them *charges*. If S and B exchange heat, S may thermalize to the canonical state; if heat and particles, the grand canonical state; and so on for electric charge, magnetization, etc. and for other thermal states. In quantum statistical mechanics, the charges are represented by Hermitian operators Q_α. The Q_α are nearly always assumed implicitly to commute with each other. For example, this assumption underlies two derivations of thermal states' forms [4, 5, 24]. But noncommutation distinguishes quantum physics from classical physics. The drive to understand nonclassicality, and to achieve quantum advantages, invites the question, what happens if the Q_α fail to commute with each other?

Jaynes and followers touched on this question during the late 20th century [25, 26]. Quantum-information thermodynamicists seized upon it recently [1–5]. Lifting the assumption that exchanged charges commute has spawned discoveries of truly quantum thermodynamics: a generalization of the microcanonical state [4], resource theories [2–4, 14–16], a generalization of the majorization preorder [9], low entropy production [13], and reference frames [10, 21] in which noncommutation plays a key role. Noncommuting charges have proven fruitful in quantum-information thermodynamics.

Most of these discoveries have been abstract, mathematical, and information-theoretic. They merit testing and further exploration in experiments. Condensed matter and atomic, molecular, and optical (AMO) physics offer potential testbeds [27, 28]. Furthermore, condensed matter, AMO physics, and high-energy physics have been incubating toolkits for studying quantum many-body thermalization (e.g., [29–37]). These toolkits call for generalizing to accommodate noncommuting charges.

A bridge for noncommuting charges, from quantum-information thermodynamics to many-body physics, has just begun to be built: First, an experiment was recently proposed for observing a qubit chain equilibrate via exchanges of spin components [12]. Second, Refs. [13, 17] analyze the exchange of energy and squeezing by bosons. Third, noncommuting charges determine the equilibration of spinless fermions simulated in [18].

This paper introduces a systematic approach to the physical realization of noncommuting charges in thermodynamics: an algorithm for constructing Hamiltonians that overtly move noncommuting charges between subsystems while conserving the charges globally. The charges form a finite-dimensional semisimple complex Lie algebra. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. Our algorithm also produces a convenient basis for the algebra—a basis of charges explicitly transported locally, and conserved globally, by the Hamiltonian. Such

* nicoleyh@g.harvard.edu
† smajidy@uwaterloo.ca

Hamiltonians can be realized with superconducting circuits, with ultracold atoms, with trapped ions, and in quantum chromodynamics (QCD).

This paper generalizes, and highlights the mathematical structures supporting, work in high-energy and nuclear physics. There, special cases of our Hamiltonians (and Lagrangian versions thereof) have been constructed to be SU(D)-symmetric [38–45].[1] Our work generalizes beyond SU(D) to every finite-dimensional semisimple complex Lie group on whose algebra the Killing form (a function reviewed below) induces a metric, or distance measure. Furthermore, we prove that these constructions work due to the roles played by the Killing form and Cartan-Weyl bases in Lie algebras.

If the charges form the Lie algebra $\mathfrak{su}(D)$, N identical subsystems form the global system, and each subsystem corresponds to the Hilbert space $\mathbb{C}^D$, then Schur-Weyl duality governs the Hamiltonians' forms [48, 49]: Some of the identical subsystems can form a system of interest, and some can form a bath. The Hamiltonians must be linear combinations of the permutations of N objects. Since Schur-Weyl duality dictates the Hamiltonians' forms, why construct Hamiltonians according to our algorithm? How to implement an arbitrary linear combination of permutations is not obvious. In contrast, our Hamiltonians have clear physical interpretations, being constructed from ladder operators: Our Hamiltonians transport noncommuting charges manifestly. Furthermore, the Schur-Weyl duality applies only if each subsystem Hilbert space is $\mathbb{C}^D$. Our results are not restricted so.

This paper begins with our setup, detailed in Sec. I. Section II introduces the Hamiltonian-construction algorithm pedagogically. We also review mathematical background and illustrate with the Lie algebra $\mathfrak{su}(2)$. Section III synopsizes the algorithm, crystallizing the main result, and presents two properties of the algorithm. A richer example provides intuition in Sec. IV: Hamiltonians that transport and conserve charges in the Lie algebra $\mathfrak{su}(3)$. Section V concludes with potential realizations of our Hamiltonians in condensed matter, AMO, and high-energy and nuclear physics.

I. SETUP

Consider a global closed quantum many-body system, as in recent thermalization experiments [50–61]. As in conventional statistical mechanics, the global system is an ensemble of N identical subsystems. A few of the subsystems form the system of interest; and the rest, an effective bath. Each subsystem corresponds to a Hilbert space $\mathcal{H}$ of finite dimensionality d.

[1] Hamiltonians have also been engineered to have SU(D) symmetry without regard to whether noncommuting charges are transported [46, 47].

The global Hamiltonian, H^{tot}, conserves extensive charges defined as follows. Let Q_α denote a Hermitian operator defined on $\mathcal{H}$. We denote by $Q_\alpha^{(j)}$ the observable defined on the j^{th} subsystem's $\mathcal{H}$. Each global observable

$$Q_\alpha^{\text{tot}} := \sum_{j=1}^{N} Q_\alpha^{(j)} \equiv \sum_{j=1}^{N} \mathbb{1}^{\otimes(j-1)} \otimes Q_\alpha^{(j)} \otimes \mathbb{1}^{\otimes(N-j)} \quad (1)$$

is conserved:

$$[H^{\text{tot}}, Q_\alpha^{\text{tot}}] = 0. \quad (2)$$

Although the local $Q_\alpha^{(j)}$ are not conserved, we will sometimes call them, and the Q_α, "charges" for convenience. One might know, initially, of only c' charges' existence.

These c' Q_α's generate a complex Lie algebra $\mathcal{A}$, which we assume to be finite-dimensional. $\mathcal{A}$ consists of all the charges (as well as non-Hermitian operators, which we ignore). Lie algebras describe many conserved physical quantities: particle number, angular momentum, electric charge, color charge, weak isospin, and our space-time's metric [49, 62, 63]. We focus on non-Abelian Lie algebras, motivated by quantum thermodynamics that highlights noncommutation: The commutator exemplifies the Lie bracket, $[Q_\alpha, Q_\beta]$.

We assume four more properties of the algebra, to facilitate our proofs. $\mathcal{A}$ is finite-dimensional and semisimple. Representing an observable, $\mathcal{A}$ is over the complex numbers. Also, on $\mathcal{A}$ is defined a Killing form (reviewed below) that induces a metric. Many physically significant algebras satisfy these assumptions—for example, the simple Lie algebras (see App. A and [49, 62, 63]).

II. PEDAGOGICAL EXPLANATION

This section describes the algorithm for constructing Hamiltonians H^{tot} that conserve noncommuting charges globally [Eq. (2)] while transporting them locally:

$$\left[H^{\text{tot}}, Q_\alpha^{(j)}\right] \neq 0 \quad (3)$$

for some site j. (In every such commutator throughout this paper, one argument implicitly contains tensor factors of $\mathbb{1}$, so that both arguments operate on the same Hilbert space.) We construct two-body interaction terms, then combine them into many-body terms. This explanation provides a pedagogical introduction; the algorithm is synopsized in Sec. III. Here, we illustrate each step with the algebra $\mathfrak{su}(2)$, which represents spin-1/2 angular momentum.

Table I lists the simple Lie algebras. Every Cartesian product of simple Lie algebras yields a semisimple Lie algebra $\mathcal{A}$. Such an algebra generates a semisimple Lie group $\mathcal{G}$. For example, if $\mathcal{A}$ consists of angular momentum, $\mathcal{A} = \mathfrak{su}(D)$. The corresponding $\mathcal{G}$ consists of rotations: $\mathcal{G} = \text{SU}(D)$.

Algebra	Dimension (c)	Rank (r)	c/r
$\mathfrak{so}(2D)$	$D(2D-1)$	D	$2D-1$
$\mathfrak{sl}(D+1)$	$(D+1)^2-1$	D	$D+2$
$\mathfrak{so}(2D+1)$	$D(2D+1)$	D	$2D+1$
$\mathfrak{sp}(2D)$	$D(2D+1)$	D	$2D+1$
$\mathfrak{g}_2$	14	2	7
$\mathfrak{f}_4$	52	4	13
$\mathfrak{e}_6$	78	6	13
$\mathfrak{e}_7$	133	7	19
$\mathfrak{e}_8$	248	8	31

Table I: Simple Lie algebras: c denotes an algebra's dimension, and r denotes the rank. We implicitly omit $\mathfrak{so}(2)$ and $\mathfrak{so}(4)$, which are not simple [63]. $\mathfrak{su}(D)$ is included in the $\mathfrak{sl}(D+1)$ entry, loosely speaking.

An algebra has two relevant properties, a dimension and a rank (Table I). The dimension, c, equals the number of generators in a basis for the algebra.[2] For example, $\mathfrak{su}(2)$ has the Pauli-operator basis $\{\sigma_x, \sigma_y, \sigma_z\}$ and so has a dimension $c = 3$. The rank, r, has a significance that we will encounter shortly.

A representation of $\mathcal{A}$ is a Lie-bracket-preserving map from $\mathcal{A}$ to a set of linear transformations. The adjoint representation maps from $\mathcal{A}$ to linear transformations defined on $\mathcal{A}$. If $x \in \mathcal{A}$, the adjoint representation $\mathrm{ad}(x)$ acts on $y \in \mathcal{A}$ as $\mathrm{ad}(x)(y) := [x, y]$. The adjoint representation features in the Killing form, which we review now. The definition of $\mathcal{A}$ involves a vector space V defined over a field F. A map $V \times V \to F$ is a *form*. The *Killing form* is the symmetric bilinear form

$$(x, y) := \mathrm{Tr}\big(\mathrm{ad}(x)\mathrm{ad}(y)\big). \tag{4}$$

We say that x and y are *Killing-orthogonal* if $(x, y) = 0$. We say that subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are Killing-orthogonal if, for all $x \in \mathcal{A}_1$ and $y \in \mathcal{A}_2$, $(x, y) = 0$. We will use the Killing form to construct the preferred basis of charges for $\mathcal{A}$.

Our construction begins with another basis: Every finite-dimensional semisimple complex Lie algebra $\mathcal{A}$ has a Cartan-Weyl basis. In fact, $\mathcal{A}$ has infinitely many. Convention may distinguish one Cartan-Weyl basis. We use the conventional $\mathfrak{su}(2)$ basis for concreteness. We use this basis, in our example, for concreteness. In general, one selects an arbitrary Cartan-Weyl basis. The basis contains generators of two types: Hermitian operators and ladder operators.

The number of Hermitian operators is the algebra's rank, r. These operators commute with each other. If $r > 1$, we rescale the operators to endow them with unit Hilbert-Schmidt norms:

$$\mathrm{Tr}\big(Q_\alpha^\dagger Q_\alpha\big) = 1. \tag{5}$$

We include these operators, $Q_{\alpha=1,2,\ldots,r}$, in our preferred basis. In the $\mathfrak{su}(2)$ example, $r = 1$; and $Q_1 = \sigma_z$, whose eigenstates $|\pm z\rangle$ correspond to the eigenvalues ± 1. The Q_α's generate a subalgebra, a *Cartan subalgebra*.

The Cartan-Weyl basis contains, as well as Hermitian operators, ladder operators. They form pairs $L_{\pm\beta}$, for $\beta = 1, 2, \ldots, \frac{c-r}{2}$. Each $L_{\pm\beta}$ raises or lowers at least one Q_α. In the $\mathfrak{su}(2)$ example, the ladder operators $\sigma_{\pm z} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ raise and lower σ_z: $L_{\pm z}|\mp z\rangle = |\pm z\rangle$. In other algebras, an $L_{\pm\beta}$ can raise and/or lower multiple Q_α's. Examples include $\mathfrak{su}(3)$ (Sec. IV).

From each ladder-operator pair, we construct an interaction that couples subsystems j and j'. Let $J_\beta^{(j,j')}$ denote a hopping frequency. An interaction that transports all the charges between j and j', while conserving each charge globally, has the form

$$H^{(j,j')} \propto \sum_{\beta=1}^{(c-r)/2} J_\beta^{(j,j')} \left(L_{+\beta}^{(j)} L_{-\beta}^{(j')} + L_{-\beta}^{(j)} L_{+\beta}^{(j')} \right). \tag{6}$$

We assemble the other terms in $H^{(j,j')}$ from other Cartan-Weyl bases, constructed as follows. Let U denote a general element of the group $\mathcal{G}$. We conjugate, with U, each element of our first Cartan-Weyl basis: For $\alpha = 1, 2, \ldots, r$ and $\beta = 1, 2, \ldots, \frac{c-r}{2}$,

$$Q_\alpha \mapsto U^\dagger Q_\alpha U = Q_{\alpha+r}, \quad \text{and} \tag{7}$$

$$L_{\pm\beta} \mapsto U^\dagger L_{\pm\beta} U = L_{\pm\left(\beta+\frac{c-r}{2}\right)}. \tag{8}$$

We include the new Q_α's (for which $\alpha = r+1, r+2, \ldots, 2r$) in our preferred basis for the algebra.

We constrain U such that each new Q_α is Killing-orthogonal to (i) each other new charge Q_β and (ii) each original charge Q_γ:

$$(Q_\alpha, Q_\beta) = (Q_\alpha, Q_\gamma) = 0 \tag{9}$$

for all $\alpha, \beta = r+1, r+2, \ldots, 2r$ and all $\gamma = 1, 2, \ldots, r$. This orthogonality restricts U, though not completely. The new Q_α's generate a Cartan subalgebra Killing-orthogonal to the original Cartan subalgebra. The new ladder operators contribute to the interaction:

$$H^{(j,j')} \propto \sum_{\beta=1}^{c-r} J_\beta^{(j,j')} \left(L_{+\beta}^{(j)} L_{-\beta}^{(j')} + \text{h.c.} \right). \tag{10}$$

In the $\mathfrak{su}(2)$ example, U can be represented by $\begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}$, wherein $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. The algorithm restricts U only via the Killing-orthogonality of $U^\dagger \sigma_z U$ to U. We enforce only this restriction in App. B. Here, we choose a U for pedagogical simplicity:

[2] We chose the notation c to evoke the c introduced in [5]. There, c was defined as the number of charges. As explained in the present paper's Sec. I, those charges would form a Lie algebra. Infinitely many charges would therefore exist, the c in [5] would equal infinity, and results in [5] would be impractical. We therefore define c as the Lie algebra's finite dimension.

$U = (\mathbb{1} + i\sigma_y)/\sqrt{2}$, such that $Q_{\alpha+r} = Q_2 = \sigma_x$. The new ladder operators, $\sigma_{\pm x} := \left(\frac{\mathbb{1}+i\sigma_y}{\sqrt{2}}\right)\sigma_{\pm z}\left(\frac{\mathbb{1}+i\sigma_y}{\sqrt{2}}\right)$, create and annihilate quanta of the x-component of the angular momentum. The interaction becomes

$$H^{(j,j')} \propto \sum_{\beta=z,x} J_\beta^{(j,j')}\left(\sigma_{+\beta}^{(j)}\sigma_{-\beta}^{(j')} + \text{h.c.}\right). \tag{11}$$

We repeat the foregoing steps: Write out the form of a general $U \in \mathcal{G}$. Conjugate each element of the original Cartan-Weyl basis with U. Constrain U such that the new Q_α's are orthogonal to each other and to the older Q_α's. Include the new Q_α's in our preferred basis for the algebra. Form a term, in $H^{(j,j')}$, from the new ladder operators $L_{\pm\beta}$.

Each Cartan-Weyl basis contributes r elements Q_α to the preferred basis. The basis contains c elements, so we form c/r mutually orthogonal Cartan-Weyl bases. c/r equals an integer for the finite-dimensional semisimple complex Lie algebras, according to Proposition 1 in Sec. III. Table I confirms the claim for the simple Lie algebras. Our algebra's finite dimensionality ensures that our algorithm halts. The two-body interaction is now

$$H^{(j,j')} = \sum_{\beta=1}^{\frac{c-r}{2}\cdot\frac{c}{r}} J_\beta^{(j,j')}\left(L_{+\beta}^{(j)}L_{-\beta}^{(j')} + \text{h.c.}\right). \tag{12}$$

Why is the preferred basis $\{Q_\alpha\}$ preferable? First, the basis endows the Hamiltonian with a simple physical interpretation: $H^{(j,j')}$ transports all these charges locally while conserving them globally. Second, the basis is (Killing-)orthogonal.

In the $\mathfrak{su}(2)$ example, $c/r = 3/1 = 3$. Hence we construct three Cartan-Weyl bases, using two SU(2) elements. If the first unitary was $(\mathbb{1} + i\sigma_y)/\sqrt{2}$, the second unitary is $(\mathbb{1} - i\sigma_x + i\sigma_y + i\sigma_z)/2$, to within a global phase. Consequently, $Q_3 = \sigma_y$, the preferred basis for $\mathcal{A}$ is $\{\sigma_z, \sigma_x, \sigma_y\}$, and

$$H^{(j,j')} = \sum_{\beta=x,y,z} J_\beta^{(j,j')}\left(\sigma_{+\beta}^{(j)}\sigma_{-\beta}^{(j')} + \text{h.c.}\right). \tag{13}$$

Next, we constrain the interaction to conserve every global charge:

$$[H^{(j,j')}, Q_\alpha^{\text{tot}}] = 0 \quad \forall \alpha = 1, 2, \ldots, c. \tag{14}$$

The commutation relations (14) constrain the hopping frequencies $J_\alpha^{(j,j')}$. The frequencies must equal each other in the $\mathfrak{su}(2)$ example: $J_\alpha^{(j,j')} \equiv J^{(j,j')}$ for all α. The Hamiltonian simplifies to [12]

$$H^{(j,j')} = J^{(j,j')}\vec{\sigma}^{(j)} \cdot \vec{\sigma}^{(j')}. \tag{15}$$

This Heisenberg model is known to have SU(2) symmetry and so to conserve each $\sigma_\alpha^{\text{tot}}$. But the Hamiltonian is typically written in the dot-product form (15), as

$$H^{(j,j')} \propto \sum_{\alpha=x,y,z} \sigma_\alpha^{(j)}\sigma_\alpha^{(j')}. \tag{16}$$

or in the z-biased form $H^{(j,j')} \propto 2(\sigma_{+z}^{(j)}\sigma_{-z}^{(j')} + \sigma_{-z}^{(j)}\sigma_{+z}^{(j')}) + \sigma_z^{(j)}\sigma_z^{(j')}$. None of these three forms reveals that the Heisenberg model transports noncommuting charges between subsystems. Our expression (13) and our algorithm do. In relativistic field theories, making the action manifestly Lorentz-invariant is worthwhile; analogously, making the Hamiltonian manifestly transport noncommuting charges locally, while conserving them globally, is worthwhile. Furthermore, our algorithm constructs Hamiltonians that overtly transport noncommuting charges locally and conserve the charges globally not only in this simple $\mathfrak{su}(2)$ example, but also for all finite-dimensional semisimple complex Lie algebras on which the Killing form induces a metric—including algebras for which this algorithm does not product the Heisenberg Hamiltonian. Appendix C discusses a generalization of the simple form (15).

We have constructed a two-body interaction $H^{(j,j')}$ that couples subsystems j and j'. We construct k-body terms $H^{\left(j,j',\ldots,j^{(k)}\right)}$ by multiplying two-body terms (12) together, constraining the couplings such that $[H^{\left(j,j',\ldots,j^{(k)}\right)}, Q_\alpha^{\text{tot}}] = 0$, and subtracting off any fewer-body terms that appear in the product. Section III details the formalism. In the $\mathfrak{su}(2)$ example, a three-body interaction has the form (App. B)

$$H^{(j,j',j'')} \propto H^{(j,j')}H^{(j',j'')}H^{(j'',j)} \tag{17}$$

$$\propto J^{(j,j',j'')}[(\sigma_x\sigma_y\sigma_z + \sigma_y\sigma_z\sigma_x + \sigma_z\sigma_x\sigma_y) - (\sigma_z\sigma_y\sigma_x + \sigma_x\sigma_z\sigma_y + \sigma_y\sigma_x\sigma_z)]. \tag{18}$$

wherein $J^{(j,j',j'')} \in \mathbb{R}$.

The Hamiltonian we constructed may be integrable. For example, the one-dimensional (1D) nearest-neighbor Heisenberg model is integrable [64]. Integrable Hamiltonians have featured in studies of noncommuting charges in thermodynamics [18]. But one might wish for the system to thermalize as much as possible, as is promoted by nonintegrability [32, 65]. Geometrically nonlocal couplings, many-body interactions, and multidimensional lattices tend to break integrability. Hence one can add terms $H^{(j,j')}$ and $H^{(j,j',\ldots,j^{(k)})}$ to the global Hamiltonian H^{tot}, and keep growing the lattice's dimensionality, until H^{tot} becomes nonintegrable. Nonintegrability may be diagnosed with, e.g., energy-gap statistics [32]. In the $\mathfrak{su}(2)$ example, one can break integrability by creating next-nearest-neighbor couplings or by making the global system two-dimensional [12].

III. ALGORITHM FOR CONSTRUCTING HAMILTONIANS THAT EXCHANGE NONCOMMUTING CHARGES

Here, we synopsize the algorithm elaborated on in Sec. II. Then, we present two results pertinent to the algorithm. We construct, as follows, Hamiltonians that transport noncommuting charges locally and conserve the charges globally:

1. Identify an arbitrary Cartan-Weyl basis for the algebra, $\mathcal{A}$.

2. The Cartan-Weyl basis contains r Hermitian operators that commute with each other. Scale each such operator such that it has a unit Hilbert-Schmidt norm [Eq. (5)]. Label the results $Q_{\alpha=1,2,\ldots,r}$. Include them in the preferred basis for the algebra.

3. The other Cartan-Weyl-basis elements are ladder operators that form raising-and-lowering pairs: $L_{\pm\beta}$, for $\beta = 1, 2, \ldots c-r$. From each pair, form one term in the two-body interaction, $H^{(j,j')}$ [Eq. (6)].

4. Write out the form of the most general element $U \in \mathcal{G}$ of the Lie group $\mathcal{G}$ generated by $\mathcal{A}$. Conjugate each charge Q_α and each ladder operator $L_{\pm\beta}$ with U [Eq. (7)]. The new charges and new ladder operators, together, form another Cartan-Weyl basis.

5. Constrain U such that every new charge Q_α is Killing-orthogonal to (i) each other new charge and (ii) each charge already in the basis [Eq. (9)].

6. Include each new Q_α in the basis for $\mathcal{A}$.

7. From each new pair $L_{\pm\beta}$ of ladder operators, form a term in the two-body interaction $H^{(j,j')}$ [Eq. (10)].

8. Repeat steps 4-7 until having identified c/r Cartan-Weyl bases, wherein c denotes the algebra's dimension. Each Cartan-Weyl basis contributes r elements Q_α to the preferred basis for $\mathcal{A}$. The basis is complete, containing $r \cdot \frac{c}{r} = c$ elements.

9. Constrain the two-body interaction to conserve each global charge [Eq. (14)], for all $\alpha = 1, 2, \ldots, c$. Solve for the frequencies $J_\beta^{(j,j')}$ that satisfy this constraint.

10. If a k-body interaction is desired, for any $k > 2$: Perform the following substeps for $\ell = 3, 4, \ldots, k$: Multiply together ℓ unconstrained two-body interactions (12) cyclically:
$$H^{(j,j',\ldots,j^{(\ell)})} = H^{(j,j')} H^{(j',j'')} \ldots H^{(j^{(\ell-1)},j^{(\ell)})} \times H^{(j^{(\ell)},j)}. \tag{19}$$
Constrain the couplings so that $[H^{(j,j',\ldots,j^{(\ell)})}, Q_\alpha^{\text{tot}}] = 0$ for all α. If $H^{(j,j',\ldots,j^{(\ell)})}$ contains fewer-body terms that conserve all the Q_α^{tot}, subtract those terms off.

11. Sum the accumulated interactions $H^{(j,j',\ldots,j^{(k)})}$ over the subsystems $j, j', \ldots$ to form H^{tot}.

12. If H^{tot} is to be nonintegrable, add longer-range interactions and/or large-k k-body interactions until breaking integrability, as signaled by, e.g., energy-gap statistics.

Having synopsized our algorithm, we present two properties of it. The first property ensures that the algorithm runs for an integer number of iterations (step 8).

Proposition 1. *Consider any finite-dimensional semisimple complex Lie algebra. The algebra's dimension, c, and rank, r, form an integer ratio: $c/r \in \mathbb{Z}_{>0}$.*

We prove this proposition in App. D. The second property characterizes the algorithm's output.

Theorem 1. *The charges $Q_1, Q_2, \ldots, Q_c$ produced by the algorithm form a basis for the algebra $\mathcal{A}$.*

Proof. The charges are Killing-orthogonal by construction: $(Q_\alpha, Q_\beta) = 0$ for all α, β. The Killing form induces a metric on $\mathcal{A}$ by assumption. Therefore, the Q_α are linearly independent according to this metric.

The algorithm produces c charges (step 8). c denotes the algebra's dimension, the number of elements in each basis for $\mathcal{A}$. Hence every linearly independent set of c $\mathcal{A}$ elements forms a basis for $\mathcal{A}$. Hence the Q_α form a basis. □

IV. $\mathfrak{su}(3)$ EXAMPLE

Section II illustrated the Hamiltonian-construction algorithm with the algebra $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$ example offered simplicity but lacks other algebras' richness: In other algebras, each Cartan-Weyl basis contains multiple Hermitian operators and multiple ladder-operator pairs. We demonstrate how our algorithm accommodates this richness, by constructing a two-body Hamiltonian that transports $\mathfrak{su}(3)$ elements locally while conserving them globally. Such Hamiltonians may be engineered for superconducting qutrits, as sketched in Sec. V. Furthermore, the Nambu–Jona-Lasinio (NJL) model, a precursor to QCD, has a similar form [67, 68]. However, this $\mathfrak{su}(3)$ example only illustrates our more general algorithm, which works for all finite-dimensional semisimple complex Lie algebras on which the Killing form induces a metric.

Each basis for $\mathfrak{su}(3)$ contains $c = 8$ elements. The most famous basis consists of the Gell-mann matrices, $\lambda_{k=1,2,\ldots,8}$ [69]. The λ_k generalize the Pauli matrices in certain ways, being traceless and Killing-orthogonal. From the Gell-mann matrices is constructed the conventional Cartan-Weyl basis [70], reviewed in App. E. The $r = 2$ Hermitian elements are Gell-mann matrices:
$$Q_1 = \lambda_3, \quad \text{and} \quad Q_2 = \lambda_8. \tag{20}$$
Q_1 and Q_2 belong in the preferred basis of charges for $\mathfrak{su}(3)$. For pedagogical clarity, we will identify all the charges before addressing the ladder operators.

A general element $U \in \mathrm{SU}(3)$ contains eight real parameters. In the Euler parameterization [71],

$$U = e^{i\lambda_3\phi_1/2} e^{i\lambda_2\phi_2/2} e^{i\lambda_3\phi_3/2} e^{i\lambda_5\phi_4/2} \times e^{i\lambda_3\phi_5/2} e^{i\lambda_2\phi_6/2} e^{i\lambda_3\phi_7/2} e^{i\lambda_8\phi_8/2} . \tag{21}$$

The parameters $\phi_1, \phi_3, \phi_5, \phi_7 \in [0, 2\pi)$; $\phi_2, \phi_4, \phi_6 \in [0, \pi]$; and $\phi_8 \in [0, 2\sqrt{3}\pi)$. We now constrain U, identifying the instances U_{i} that map the first charges to $Q_3 = U_{\mathrm{i}}^\dagger Q_1 U_{\mathrm{i}}$ and $Q_4 = U_{\mathrm{ii}}^\dagger Q_2 U_{\mathrm{ii}}$ that are Killing-orthogonal to each other and to the original charges. Appendix E contains the details. We label with a superscript (i) the parameters used to fix U_{i}: $\phi_1^{(\mathrm{i})}$, $\phi_3^{(\mathrm{i})}$, $\phi_7^{(\mathrm{i})}$, $\phi_8^{(\mathrm{i})}$, and $n^{(\mathrm{i})}$. For convenience, we package several parameters together: $a^{(\mathrm{i})} := \frac{1}{2}\left(\phi_3^{(\mathrm{i})} - \phi_7^{(\mathrm{i})} - \sqrt{3}\phi_8^{(\mathrm{i})} + \pi n^{(\mathrm{i})} + \frac{\pi}{2}\right)$, and $b^{(\mathrm{i})} := a^{(\mathrm{i})} + \phi_7^{(\mathrm{i})}$. In terms of these parameters, the new charges have the forms (App. E)

$$\begin{aligned} Q_3 = \frac{1}{\sqrt{3}} \Big[&(-1)^{n^{(\mathrm{i})}+1} \sin\Big(a^{(\mathrm{i})} - b^{(\mathrm{i})}\Big) \lambda_1 \\ &- (-1)^{n^{(\mathrm{i})}} \cos\Big(a^{(\mathrm{i})} - b^{(\mathrm{i})}\Big) \lambda_2 - \sin\Big(a^{(\mathrm{i})}\Big) \lambda_4 \\ &- \cos\Big(a^{(\mathrm{i})}\Big) \lambda_5 + \sin\Big(b^{(\mathrm{i})}\Big) \lambda_6 + \cos\Big(b^{(\mathrm{i})}\Big) \lambda_7 \Big] \quad \text{and} \end{aligned} \tag{22}$$

$$\begin{aligned} Q_4 = \frac{(-1)^{n^{(\mathrm{i})}}}{\sqrt{3}} \Big[&(-1)^{n^{(\mathrm{i})}+1} \cos\Big(a^{(\mathrm{i})} - b^{(\mathrm{i})}\Big) \lambda_1 \\ &+ (-1)^{n^{(\mathrm{i})}} \sin\Big(a^{(\mathrm{i})} - b^{(\mathrm{i})}\Big) \lambda_2 + \cos\Big(a^{(\mathrm{i})}\Big) \lambda_4 \\ &- \sin\Big(a^{(\mathrm{i})}\Big) \lambda_5 + \cos\Big(b^{(\mathrm{i})}\Big) \lambda_6 - \sin\Big(b^{(\mathrm{i})}\Big) \lambda_7 \Big]. \end{aligned} \tag{23}$$

Q_3 has the same form as Q_5 and Q_7, which satisfy the same Killing-orthogonality conditions. Similarly, Q_4 has the same form as Q_6 and Q_8. The later charges' parameters $a^{(\ell)}$ and $b^{(\ell)}$ are more restricted, however (App. E). We have identified our preferred basis of charges.

Let us construct the ladder operators and Hamiltonian. Each Cartan-Weyl basis contains $c - r = 8 - 2 = 6$ ladder operators. The conventional Cartan-Weyl basis contains ladder operators formed from Gell-man matrices:

$$L_{\pm 1} := \tfrac{1}{2}(\lambda_1 \pm i\lambda_2), \quad L_{\pm 2} := \tfrac{1}{2}(\lambda_4 \pm i\lambda_5), \quad \text{and } L_{\pm 3} := \tfrac{1}{2}(\lambda_6 \pm i\lambda_7). \tag{24}$$

Transforming these operators with unitaries $U_{\mathrm{ii,iii,iv}}$ yields $L_{\pm 4}$ through $L_{\pm 12}$, whose forms appear in App. E. From each ladder operator, we form one term in the two-body Hamiltonian (6).

Finally, we determine the hopping frequencies $J_\alpha^{(j,j')}$, demanding that $[H^{(j,j')}, Q_\alpha^{\mathrm{tot}}] = 0$ for all α. For all possible values of the $a^{(\ell)}$, $b^{(\ell)}$, and $n^{(\ell)}$, if all the frequencies are nonzero, then all the frequencies equal each other. We set $J_\alpha^{(j,j')} \equiv \frac{4}{3} J^{(j,j')}$, such that

$$H^{(j,j')} = J^{(j,j')} \sum_{\alpha=1}^{8} \lambda_\alpha^{(j)} \lambda_\alpha^{(j')} \propto \sum_{\alpha=1}^{8} Q_\alpha^{(j)} Q_\alpha^{(j')}. \tag{25}$$

The Hamiltonian collapses to a simple form analogous to the $\mathfrak{su}(2)$ example's Eq. (16) [see Eq. (C1) in App. C].

V. OUTLOOK

We have presented an algorithm for constructing Hamiltonians that transport noncommuting charges locally while conserving the charges globally. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. The algorithm produces, as well as Hamiltonians, preferred bases of charges that are (i) overtly transported locally and conserved globally and (ii) Killing-form-orthogonal. This construction works whenever the charges form a finite-dimensional semisimple complex Lie algebra on which the Killing form induces a metric. We therefore generalize, and provide a broad mathematical foundation for, more-specific constructions in high-energy and nuclear physics.

This work provides a systematic means of bridging noncommuting thermodynamic charges from abstract quantum information theory to condensed matter, AMO physics, and high-energy and nuclear physics. The mathematical results that have accrued [1–23] can now be tested experimentally, via our construction. Our Hamiltonians can be realized in a variety of platforms, which we now discuss.

As mentioned in Sec. IV, the NJL model acts similarly to the Hamiltonian (25) constructed from $\mathfrak{su}(3)$. The NJL model provides an effective theory for QCD, the simulation of which on quantum computers was proposed recently [44, 72, 73]: Simulating QCD requires so many computational resources, a quantum speedup is hoped for. Contrarily, thermodynamics excels at simple representations. The present work may enable the quantum thermodynamics of noncommuting charges to simplify some QCD problems.

Second, the Heisenberg model (13) can be implemented with ultracold atoms and trapped ions [74–79]. Reference [12] details how to harness these setups to study noncommuting thermodynamic charges.

Finally, superconducting circuits can serve as qudits with Hilbert-space dimensionalities $d \geq 2$ [80]. Qutrits have been realized with transmons, slightly anharmonic oscillators [81]. The lowest two energy levels often serve as a qubit, but the second energy gap nearly equals the first. Hence the third level can be addressed relatively easily [82]. Superconducting qutrits offer a tabletop platform for transporting and conserving $\mathfrak{su}(3)$ charges as in Sec. IV.

Experiments ≤ 5 qubits have been run [83, 84], Furthermore, many of the tools used to control and measure superconducting qubits can be applied to qutrits [82, 85–94]. A noncommuting-charges-in-thermodynamics experiment may begin with preparing the qutrits in an approximate microcanonical subspace, a generalization of the microcanonical subspace that accommodates noncommuting charges [5]. Such a state preparation may be achieved with weak measurements [12], which have been performed on superconducting qudits through cavity quantum electrodynamics [95].

T_2^* relaxation times of $\sim 39\ \mu$s, for the lowest energy gap, and ~ 14 μs, for the second-lowest gap, have been achieved [84]. Meanwhile, two-qutrit gates can be realized in $\sim 10 - 10^2$ ns [84, 96, 97]. Some constant number of such gates may implement one three-level gate that simulates a term in our Hamiltonian. If the number is order-10, information should be able to traverse an 8-qutrit system ~ 10 times before the qutrits decohere detrimentally. Simulations of our Hamiltonians should therefore be able to thermalize the system internally. The states of small subsystems, such as qutrit pairs, can be read out via quantum state tomography [82, 85–88]. Hence superconducting qutrits, and other platforms, can import noncommuting charges from quantum thermodynamics to many-body physics, by simulating the Hamiltonians constructed here.

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Appendix A THE KILLING FORM INDUCES A METRIC ON EVERY SIMPLE LIE ALGEBRA.

Here, we prove a claim made in Sec. I: The Killing form induces a metric on every simple Lie algebra. The proof relies on background material reviewed in Sec. II.

Every inner product defines a metric. Therefore, proving that the Killing form induces an inner product suffices. On a simple Lie algebra, all symmetric bilinear forms equal each other to within a multiplicative constant [98]. The Killing form is one symmetric bilinear form; another is $\text{Tr}(Q_\alpha Q_\beta)$. Hence $(Q_\alpha, Q_\beta) \propto \text{Tr}(Q_\alpha Q_\beta) = \text{Tr}\big(Q_\alpha^\dagger Q_\beta\big)$. The final equality follows from the charges' Hermiticity. The final expression is the Hilbert-Schmidt inner product. Hence the Killing form induces an inner product.

Appendix B GENERAL HAMILTONIAN THAT TRANSPORTS $\mathfrak{su}(2)$ ELEMENTS LOCALLY WHILE CONSERVING THEM GLOBALLY

Section II illustrated how to construct Hamiltonians that transport $\mathfrak{su}(2)$ elements locally while conserving them globally. The illustration was not maximally general; we restricted a unitary U more than required, for pedagogy. We generalize the construction here. For clarity of presentation, we derive the charges' forms first (App. B 1) and the ladder operators' forms second (App. B 2). We then construct the two-body Hamiltonian $H^{(j,j')}$ and a three-body Hamiltonian (App. B 3).

B 1 Preferred basis of charges for $\mathfrak{su}(2)$

The conventional Cartan-Weyl basis contains the Hermitian operator

$$Q_1 = \sigma_z. \tag{B1}$$

To identify the next Cartan-Weyl basis, we invoke a general unitary $U \in \text{SU}(2)$. In the Euler parameterization,

$$U = e^{i\sigma_z \phi_1/2} e^{i\sigma_y \phi_2/2} e^{i\sigma_z \phi_3/2}, \tag{B2}$$

wherein $\phi_1 \in [0, 2\pi)$, $\phi_2 \in [0, \pi]$, and $\phi_3 \in [0, 2\pi)$. We restrict this general unitary to a U_i that maps Q_2 to a Killing-orthogonal charge $Q_2 = U_\text{i}^\dagger Q_1 U_\text{i}$. For $X, Y \in \mathfrak{su}(D)$, the Killing form evaluates to $(X, Y) = \text{Tr}(XY)$ [98]. Hence the Killing form between the charges is

$$0 = \left(U_\text{i}^\dagger Q_1 U_\text{i},\, Q_1\right) = \text{Tr}\left(U_\text{i}^\dagger Q_1 U_\text{i} Q_1\right) = 2\cos\left(\phi_2^{(\text{i})}\right). \tag{B3}$$

The superscript (i), here and below, labels a parameter as belonging to U_i. The equation, with $\phi_2^{(\text{i})} \in [0, \pi]$, implies that $\phi_2^{(\text{i})} = \pi/2$. The unitary and charge assume the forms

$$U_\text{i} = e^{i\sigma_z \phi_1^{(\text{i})}/2} e^{i\sigma_y \pi/4} e^{i\sigma_z \phi_3^{(\text{i})}/2} \quad \text{and} \quad Q_2 = \cos\Big(\phi_3^{(\text{i})}\Big)\sigma_x + \sin\Big(\phi_3^{(\text{i})}\Big)\sigma_y. \tag{B4}$$

Having identified the second charge, we identify the final one. We transform Q_1 with a unitary $U_{\rm ii} \in \mathrm{SU}(2)$ such that $Q_3 = U_{\rm ii}^\dagger Q_1 U_{\rm ii}$ is Killing-orthogonal to the first two charges. The first orthogonality constraint has the form of Eq. (B3), except that a (ii) replaces the superscript (i). The second orthogonality constraint is

$$0 = \mathrm{Tr}\left(U_{\rm ii}^\dagger Q_1 U_{\rm ii},\, Q_2\right) = \mathrm{Tr}\left(U_{\rm ii}^\dagger Q_1 U_{\rm ii} Q_2\right) = 2\cos\left(\phi_3^{(\rm i)} - \phi_3^{(\rm ii)}\right). \tag{B5}$$

Hence $\phi_3^{(\rm ii)} = \phi_3^{(\rm i)} + \pi\left(n^{(\rm ii)} - \frac{1}{2}\right)$, wherein $n^{(\rm ii)} \in \mathbb{Z}$. Hence $U_{\rm ii}$ and Q_3 have the forms

$$U_{\rm ii} = e^{i\sigma_z \phi_1^{(\rm ii)}/2} e^{i\sigma_y \pi/4} e^{i\sigma_z[\phi_3^{(\rm i)} + \pi(n - \frac{1}{2})]/2} \quad \text{and} \tag{B6}$$

$$Q_3 = (-1)^{n^{(\rm ii)}}\left[\sin\left(\phi_3^{(\rm i)}\right)\sigma_x - \cos\left(\phi_3^{(\rm i)}\right)\sigma_y\right]. \tag{B7}$$

Equations (B7), (B4), and (B1) specify the preferred basis of charges for $\mathfrak{su}(2)$.

B 2 General ladder operators for $\mathfrak{su}(2)$

The conventional Cartan-Weyl basis contains operators that raise and lower σ_z:

$$L_{\pm 1} = \sigma_{\pm z} = \frac{1}{2}(\sigma_x \pm i\sigma_y). \tag{B8}$$

Conjugation with $U_{\rm i}$ yields the ladder operators for Q_2, and conjugation with $U_{\rm ii}$ yields the ladder operators for Q_3:

$$L_{\pm 2} = U_{\rm i}^\dagger L_{\pm 1} U_{\rm i} = \frac{-e^{\mp i\phi_1^{(\rm i)}}}{2}[\sigma_z \pm i(\sin\{\phi_3^{(\rm i)}\}\sigma_x - \cos\{\phi_3^{(\rm i)}\}\sigma_y)], \quad \text{and} \tag{B9}$$

$$L_{\pm 3} = U_{\rm ii}^\dagger L_{\pm 1} U_{\rm ii} = \frac{-e^{\mp i\phi_1^{(\rm ii)}}}{2}\left\{\sigma_z \mp i(-1)^{n^{(\rm ii)}}\left[\cos\left(\phi_3^{(\rm i)}\right)\sigma_x + \sin\left(\phi_3^{(\rm i)}\right)\sigma_y\right]\right\}. \tag{B10}$$

B 3 Two-body and three-body Hamiltonians for $\mathfrak{su}(2)$

To form $H^{(j,j')}$, we substitute for the ladder operators from Eqs. (B8) and (B9) into Eq. (12). We require that $H^{(j,j')}$ conserve each global charge, imposing Eq. (14). This equation holds, algebra reveals, if and only if the hopping frequencies $J_\alpha^{(j,j')}$ equal each other. The Hamiltonian simplifies to Eq. (15). The final expression does not depend on our choice of $\phi_k^{(\rm i)}$, $\phi_k^{(\rm ii)}$, or $n^{(\rm i)}$.

Let us construct a Hamiltonian $H^{(j,j',j'')}$ that transfers $\mathfrak{su}(2)$ charges between three sites—j, j', and j''—while conserving the charges globally. We multiply three two-body Hamiltonians together cyclically:

$$H^{(j,j',j'')} \propto H^{(j,j')} H^{(j',j'')} H^{(j'',j)} \tag{B11}$$

We substitute in from Eq. (13), the $H^{(j,j')}$ expression in which the hopping frequencies have not yet been restricted. The frequencies can assume different values, when $[H^{(j,j',j'')},\, Q_\alpha^{\rm tot}] = 0$, than when $[H^{(j,j')},\, Q_\alpha^{\rm tot}] = 0$. Imposing the first commutator equation yields four sets of solutions for the J_α's, when $J_\alpha \neq 0$ for all α:

1. $J_1 = J_2 = J_3$, $J_4 = J_5 = J_6$, and $J_7 = J_8 = J_9$.
2. $J_1 = J_2 = -J_3$, $J_4 = J_5 = -J_6$, and $J_7 = J_8 = -J_9$.
3. $J_1 = J_2 = \frac{-J_3}{2}$, $J_4 = J_5 = \frac{-J_6}{2}$, and $J_7 = J_8 = \frac{-J_9}{2}$.
4. $\frac{J_2}{J_1} = \frac{J_5}{J_4} = \frac{J_8}{J_7}$, $J_1 + J_2 = -J_3$, $J_4 + J_5 = -J_6$, and $J_7 + J_8 = -J_9$.

We have omitted superscripts for conciseness. The four solutions lead to distinct Hamiltonians.[3]

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[3] However, each solution contains a little redundancy: Consider picking one of the four solutions, then cycling the indices in $(1, 2, 3)$ identically to the indices in $(4, 5, 6)$ and to the indices in $(7, 8, 9)$. The resulting J_α's specify a Hamiltonian identical to the original.

For concreteness, we detail the first set of solutions, item 1. We collect three of the frequencies to simplify notation: $J^{j,j',j''} = J_1^{(j,j')} J_4^{(j',j'')} J_7^{(j',j'')}$. Substituting the J_α's into the Hamiltonian (B11) yields

$$H^{(j,j',j'')} \propto J^{(j,j',j'')} \Big\{ 3\mathbb{1}\mathbb{1}\mathbb{1} - 2 \left(H^{(j,j')} - H^{(j'',j)} + H^{(j',j'')} \right) + i[(\sigma_x\sigma_y\sigma_z + \sigma_y\sigma_z\sigma_x + \sigma_z\sigma_x\sigma_y) - (\sigma_z\sigma_y\sigma_x + \sigma_x\sigma_z\sigma_y + \sigma_y\sigma_x\sigma_z)\}. \tag{B12}$$

We have omitted some superscripts to simplify notation. The first term is trivial, terms 2-4 are two-body, and each of terms 1-4 conserves each Q_α^{tot}. Subtracting these terms off yields the solely three-body Hamiltonian (18). We have absorbed the i into the coefficient such that $J^{j,j',j''} \in \mathbb{R}$.

Appendix C SIMPLE FORM TO WHICH A TWO-BODY HAMILTONIAN MAY COLLAPSE

In the $\mathfrak{su}(2)$ example, $H^{(j,j')}$ collapsed to the simple form (16). The $\mathfrak{su}(3)$ $H^{(j,j')}$ collapses to an analogous form, we shown in Sec. IV. This form generalizes to

$$\sum_{\alpha=1}^{c} Q_\alpha^{(j)} Q_\alpha^{(j')}. \tag{C1}$$

This expression generally conserves noncommuting charges globally, and transport the charges locally, as proved below. However, the expression's equality with a two-body Hamiltonian that clearly, overtly transports local charges from site to site is proved only in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ examples.

Proposition 2. *Consider any Lie algebra whose structure constants have the antisymmetry property*

$$f_{\alpha\beta}^{\gamma} = -f_{\gamma\beta}^{\alpha}. \tag{C2}$$

A two-body Hamiltonian of the form (C1) *conserves the algebra's elements globally.*

Every compact semisimple Lie algebra has such structure constants [66].

Proof. First, we substitute from Eq. (C1) into the conservation law. Then, we invoke the commutator's linearity and the arguments' tensor-product forms:

$$0 = \left[H^{(j,j')}, Q_\alpha^{\text{tot}} \right] = \left[\sum_{\beta=1}^{c} Q_\beta^{(j)} Q_\beta^{(j')}, Q_\alpha^{(j)} \otimes \mathbb{1}^{(j')} + \mathbb{1}^{(j)} \otimes Q_\alpha^{(j')} \right] \tag{C3}$$

$$= \sum_{\beta=1}^{c} \left(\left[Q_\beta^{(j)} Q_\beta^{(j')}, Q_\alpha^{(j)} \otimes \mathbb{1}^{(j')} \right] + \left[Q_\beta^{(j)} Q_\beta^{(j')}, \mathbb{1}^{(j)} \otimes Q_\alpha^{(j')} \right] \right) \tag{C4}$$

$$= \sum_{\beta=1}^{c} \left(\left[Q_\beta^{(j)}, Q_\alpha^{(j)} \right] Q_\beta^{(j')} + Q_\beta^{(j)} \left[Q_\beta^{(j')}, Q_\alpha^{(j')} \right] \right). \tag{C5}$$

Let $f_{\alpha\beta}^{\gamma}$ denote the Lie algebra's structure constants. The f's dictate how a Lie bracket decomposes as a linear combination of the algebra's elements:

$$[Q_\alpha, Q_\beta] = \sum_{\gamma=1}^{c} f_{\alpha\beta}^{\gamma} Q_\gamma. \tag{C6}$$

We substitute into Eq. (C5), then pull the sums and constants out front:

$$0 = \sum_{\beta=1}^{c} \left[\left(\sum_{\gamma=1}^{c} f_{\beta\alpha}^{\gamma} Q_\gamma^{(j)} \right) Q_\beta^{(j')} + Q_\beta^{(j)} \left(\sum_{\gamma=1}^{c} f_{\beta\alpha}^{\gamma} Q_\gamma^{(j')} \right) \right] = \sum_{\beta,\gamma=1}^{c} f_{\beta\alpha}^{\gamma} \left(Q_\gamma^{(j)} Q_\beta^{(j')} + Q_\beta^{(j)} Q_\gamma^{(j')} \right). \tag{C7}$$

The final equation holds if $f_{\beta\alpha}^{\gamma} = -f_{\gamma\alpha}^{\beta}$. Consider relabeling the index α as β and vice versa. Equation (C2) results. □

Having proved that the simple operator (C1) conserves noncommuting charges globally, we prove that it transports charges locally.

Proposition 3. *The simple two-body Hamiltonian* (C1) *transports the charges* Q_α *locally.*

Proof. Charge Q_α is transported locally if it satisfies Eq. (3), having a nonzero commutator

$$\left[H^{(j,j')}, Q_\alpha^{(j)}\right] = \left[\sum_{\beta=1}^{c} Q_\beta^{(j)} Q_\beta^{(j')}, Q_\alpha^{(j)}\right] = \sum_{\beta=1}^{c} \left[Q_\beta^{(j)}, Q_\alpha^{(j)}\right] Q_\beta^{(j')} = \sum_{\beta,\gamma=1}^{c} f_{\beta\alpha}^{\gamma}\, Q_\gamma^{(j)} Q_\beta^{(j')}. \tag{C8}$$

The final expression vanishes if Q_α commutes with all the other charges Q_γ in the preferred basis. If a Lie algebra has a basis of which one element commutes with the others, the algebra is Abelian, by definition [98]. We assume that the algebra $\mathcal{A}$ is non-Abelian (Sec. I). Therefore, the right-hand side of (C8) is nonzero, and the Hamiltonian transports the charges locally. □

Appendix D PROOF OF PROPOSITION 1

Proposition 1 states that the algebra $\mathcal{A}$ has an integer ratio c/r, wherein c denotes the algebra's dimension and r denotes the rank.

Proof. For every finite-dimensional complex Lie algebra, there exists a corresponding connected Lie group that is unique to within finite coverings. The Lie algebra has the same dimension and rank as each of the corresponding Lie groups. Thus, if Proposition 1 holds for all semisimple Lie groups, it holds for all semisimple Lie algebras. We prove the group claim.

Every Lie group has a maximal torus $\mathbb{T}^r$, which is the group generated by a Cartan subalgebra of the Lie algebra. The torus's dimensionality equals the group's rank, r. A torus is an r-fold Cartesian product of $\mathbb{S}^1$ manifolds [equivalently, of the group U(1)]. Quotienting out the torus's action from the Lie group yields a finite-dimensional coset space. Every finite-dimensional coset space's dimensionality is a positive integer $n \in \mathbb{Z}_{>0}$. Thus, the semisimple Lie group's dimension is $c = rn$. □

Appendix E MATHEMATICAL DETAILS: CONSTRUCTION OF A TWO-BODY HAMILTONIAN THAT TRANSPORTS $\mathfrak{su}(3)$ ELEMENTS LOCALLY WHILE CONSERVING THEM GLOBALLY

Section IV illustrated the Hamiltonian-construction algorithm with $\mathfrak{su}(3)$. We flesh out the explanation here. Appendix E 1 reviews the conventional Cartan-Weyl basis for $\mathfrak{su}(3)$. Appendix E 2 identifies the preferred basis of charges for $\mathfrak{su}(3)$. Appendix E 3 presents the ladder operators from which we construct a Hamiltonian.

E 1 Conventional Cartan-Weyl basis for $\mathfrak{su}(3)$

$\mathfrak{su}(3)$ has dimension $c = 8$ and rank $r = 2$. The conventional Cartan-subalgebra generators are denoted by $t_z = \lambda_3/2$ and $y = \lambda_8/\sqrt{3}$, wherein λ_3 and λ_8 denote Gell-mann matrices [70]. These generators, in the three-dimensional representation of $\mathfrak{su}(3)$, manifest as

$$T_z = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{E1}$$

t_z and y are orthogonal relative to the Killing form. They (more precisely, rescaled versions of them) belong in our preferred basis of charges: $Q_1 \propto t_z$, and $Q_2 \propto y$.

These charges are raised and lowered by $c - r = 8 - 2 = 6$ ladder operators, $t_\pm = (\lambda_1 \pm i\lambda_2)/2$, $v_\pm = (\lambda_4 \pm i\lambda_5)/2$, and $u_\pm = (\lambda_6 \pm i\lambda_7)/2$. In the three-dimensional representation of $\mathfrak{su}(3)$, the ladder operators manifest as

$$T_+ = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_- = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_+ = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_- = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{E2}$$

$$U_+ = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad U_- = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{E3}$$

The ladder operators participate in the following commutation relations with the charges:

$$[t_z, t_\pm] = \pm t_\pm, \quad [y, t_\pm] = 0, \tag{E4}$$

$$[t_z, v_\pm] = \pm\frac{1}{2} v_\pm, \quad [y, v_\pm] = \pm v_\pm \tag{E5}$$

$$[t_z, u_\pm] = \mp\frac{1}{2} u_\pm, \quad \text{and} \quad [y, u_\pm] = \pm u_\pm. \tag{E6}$$

These relations imply that (i) $t_\pm$ raises and lowers t_z, whereas (ii) $v_\pm$ raises or lowers both t_z and y, as does $u_\pm$. We can prove this physical significance easily: Let $L_\pm$ denote a ladder operator (a $t_\pm$, a $v_\pm$, or a $u_\pm$) that raises/lowers a charge Q. Let $|\psi\rangle$ denote a Q eigenstate associated with the eigenvalue q: $Q|\psi\rangle = q|\psi\rangle$. Consider operating on the state with the ladder operator: $L_\pm |\psi\rangle$. Suppose, for notational convenience, that, (i) if L_+ operates, q is not the greatest Q eigenvalue and (ii) if L_- operates, q is not the least Q eigenvalue. The resulting state is a Q eigenstate associated with the eigenvalue $q \pm a$, wherein $a = 1$ or $1/2$. To prove this claim, we operate on the new state with the charge: $Q(L_\pm |\psi\rangle)$. Invoking the appropriate commutation relation [Eqs. (E4)-(E6)] yields

$$QL_\pm |\psi\rangle = (L_\pm Q \pm L_\pm)|\psi\rangle = L_\pm(Q \pm a\mathbb{1})|\psi\rangle = L_\pm(q \pm a)|\psi\rangle = (q \pm a)L_\pm|\psi\rangle. \tag{E7}$$

By Eqs. (E4)-(E7), $t_\pm$ raises/lowers the t_z charge by one quantum and preserves y. $u_\pm$ lowers/raises t_z by half a quantum and raises/lowers y by one quantum. $v_\pm$ raises/lowers each of t_z and y by one quantum.

Having reviewed the conventional Cartan-Weyl basis for $\mathfrak{su}(3)$, we dispense with the conventional notation (t_z, $t_\pm$, etc.). We revert to the notation introduced in the main text (Q_α and $L_{\pm\alpha}$).

E 2 Preferred basis of charges for $\mathfrak{su}(3)$

The first two charges appear in Eqs. (20). We construct two new charges from Q_1, Q_2, and a unitary $U \in \mathrm{SU}(2)$. The general form of such a U, appears, in the Euler parameterization, in Eq. (21). We constrain U with the Killing-orthogonality conditions (9), obtaining a unitary U_i. The transformed charges have the forms $Q_3 = U_\mathrm{i}^\dagger Q_1 U_\mathrm{i}$ and $Q_4 = U_\mathrm{i}^\dagger Q_2 U_\mathrm{i}$. The new charges are Killing-orthogonal to each other by unitarity: $0 = \mathrm{Tr}\left(\left[U_\mathrm{i}^\dagger Q_1 U_\mathrm{i}\right]\left[U_\mathrm{i}^\dagger Q_2 U_\mathrm{i}\right]\right) = \mathrm{Tr}(Q_1 Q_2) = 0$. Killing-orthogonality to the old charges, Eq. (20), with the form of the $\mathfrak{su}(D)$ Killing form [98], implies

$$0 = \mathrm{Tr}\left(\left[U_\mathrm{i}^\dagger Q_1 U_\mathrm{i}\right] Q_2\right) = -\cos(\phi_2)/3, \qquad 0 = \mathrm{Tr}\left(\left[U_\mathrm{i}^\dagger Q_1 U_\mathrm{i}\right] Q_1\right) = -\frac{1}{2\sqrt{3}}\cos(\phi_3 + \phi_5), \tag{E8}$$

$$0 = \mathrm{Tr}\left(\left[U_\mathrm{i}^\dagger Q_2 U_\mathrm{i}\right] Q_2\right) = \frac{1}{2}\left[\cos(\phi_4) + \frac{1}{3}\right], \quad \text{and} \quad 0 = \mathrm{Tr}\left(\left[U_\mathrm{i}^\dagger Q_2 U_\mathrm{i}\right] Q_1\right) = -\cos(\phi_6)/3. \tag{E9}$$

Since $\phi_2, \phi_4, \phi_6 \in [0, \pi]$ and $\phi_3, \phi_5 \in [0, 2\pi)$, $\phi_2 = \frac{\pi}{2}$, $\phi_4 = \mathrm{acos}(-1/3)$, $\phi_6 = \frac{\pi}{2}$ and $\phi_5 = \pi(n - 1/2) - \phi_3$, for $n \in \{1, 2, 3, 4\}$.

Transforming Q_1 and Q_2 with a $U_\mathrm{ii} \in \mathrm{SU}(3)$ yields the charges Q_5 and Q_6, and transforming Q_1 and Q_2 with a $U_\mathrm{iii} \in \mathrm{SU}(3)$ yields Q_7 and Q_8. These last four charges are Killing-orthogonal to Q_1 and Q_2, like Q_3 and Q_4. So U_ii and U_iii share the form of U_i. However, parameters $a^{(\mathrm{ii})}$ and $b^{(\mathrm{ii})}$, or $a^{(\mathrm{iii})}$ and $b^{(\mathrm{iii})}$, replace the $a^{(\mathrm{i})}$ and $b^{(\mathrm{i})}$. The later unitaries' parameters are more constrained than the U_i parameters. Similarly, Q_5 through Q_8 share the forms of Q_3 and Q_1, apart from their more-constrained parameters.

Evaluating the restrictions on all the charges simultaneously will prove useful. First, the conditions for Q_5 to be orthogonal to Q_3 and Q_4 are

$$0 = \mathrm{Tr}(Q_5 Q_3) \propto (-1)^{n^{(\mathrm{i})} + n^{(\mathrm{ii})}} \cos\left(a^{(\mathrm{i})} - a^{(\mathrm{ii})} - b^{(\mathrm{i})} + b^{(\mathrm{ii})}\right) + \cos\left(a^{(\mathrm{i})} - a^{(\mathrm{ii})}\right) + \cos\left(b^{(\mathrm{i})} - b^{(\mathrm{ii})}\right) \quad \text{and} \tag{E10}$$

$$0 = \mathrm{Tr}(Q_5 Q_4) \propto (-1)^{n^{(\mathrm{i})} + n^{(\mathrm{ii})}} \sin\left(a^{(\mathrm{i})} - a^{(\mathrm{ii})} - b^{(\mathrm{i})} + b^{(\mathrm{ii})}\right) - \sin\left(a^{(\mathrm{i})} - a^{(\mathrm{ii})}\right) + \sin\left(b^{(\mathrm{i})} - b^{(\mathrm{ii})}\right). \tag{E11}$$

The orthogonality conditions for Q_6 impose the same constraints, since $\mathrm{Tr}(Q_6 Q_3) \propto \mathrm{Tr}(Q_5 Q_4)$ and $\mathrm{Tr}(Q_6 Q_4) \propto \mathrm{Tr}(Q_5 Q_3)$ (as can be checked explicitly). Similarly, the orthogonality conditions on Q_7 evaluate to

$$0 = \mathrm{Tr}(Q_7 Q_3) \propto (-1)^{n^{(\mathrm{i})} + n^{(\mathrm{iii})}} \cos\left(a^{(\mathrm{i})} - a^{(\mathrm{iii})} - b^{(\mathrm{i})} + b^{(\mathrm{iii})}\right) + \cos\left(a^{(\mathrm{i})} - a^{(\mathrm{iii})}\right) + \cos\left(b^{(\mathrm{i})} - b^{(\mathrm{iii})}\right), \tag{E12}$$

$$0 = \mathrm{Tr}(Q_7 Q_4) \propto (-1)^{n^{(\mathrm{i})} + n^{(\mathrm{iii})}} \sin\left(a^{(\mathrm{i})} - a^{(\mathrm{iii})} - b^{(\mathrm{i})} + b^{(\mathrm{iii})}\right) - \sin\left(a^{(\mathrm{i})} - a^{(\mathrm{iii})}\right) + \sin\left(b^{(\mathrm{i})} - b^{(\mathrm{iii})}\right), \tag{E13}$$

$$0 = \mathrm{Tr}(Q_7 Q_5) \propto (-1)^{n^{(\mathrm{ii})}+n^{(\mathrm{iii})}} \cos\Big(a^{(\mathrm{ii})} - a^{(\mathrm{iii})} - b^{(\mathrm{ii})} + b^{(\mathrm{iii})}\Big) + \cos\Big(a^{(\mathrm{ii})} - a^{(\mathrm{iii})}\Big) + \cos\Big(b^{(\mathrm{ii})} - b^{(\mathrm{iii})}\Big), \text{ and} \tag{E14}$$

$$0 = \mathrm{Tr}(Q_7 Q_6) \propto (-1)^{n^{(\mathrm{ii})}+n^{(\mathrm{iii})}} \sin\Big(a^{(\mathrm{ii})} - a^{(\mathrm{iii})} - b^{(\mathrm{ii})} + b^{(\mathrm{iii})}\Big) - \sin\Big(a^{(\mathrm{ii})} - a^{(\mathrm{iii})}\Big) + \sin\Big(b^{(\mathrm{ii})} - b^{(\mathrm{iii})}\Big). \tag{E15}$$

The orthogonality conditions for Q_8 impose the same constraints [Eqs. (E12)-(E15)].

We now identify sets of $a^{(\ell)}, b^{(\ell)}$, and $n^{(\ell)}$ that are solutions for all six constraints, Eqs. (E10)-(E15). First, we define $x_{\ell m} := a^{(\ell)} - a^{(m)}$ and $y_{\ell m} := b^{(\ell)} - b^{(m)}$, for $(\ell, m) = (2,3), (2,4), (3,4)$. By these definitions, $x_{24} = x_{23} + x_{34}$, and $y_{24} = y_{23} + y_{34}$. Second, the values of the $n^{(\ell)}$ themselves are irrelevant. Only whether $n^{(\ell)} + n^{(m)}$ is even or odd matters. Only four unique possibilities for the $n^{(\ell)}$ exist: All the $n^{(\ell)} + n^{(m)}$ are even; or one $n^{(\ell)} + n^{(m)}$ is even, while the other two sums are odd. A solution can therefore be expressed in terms of just four quantities: x_{23}, x_{34}, y_{23}, and y_{34}. Each solution is periodic:

$$(x_{23}, x_{34}, y_{23}, y_{34}) \equiv (x_{23}, x_{34}, y_{23}, y_{34}) + (2\pi n, 2\pi n, 2\pi n, 2\pi n), \tag{E16}$$

wherein $n \in \mathbb{Z}$. Therefore, we omit the $2\pi n$ when listing the solutions below.

First, suppose that all the $n^{(\ell)} + n^{(m)}$ are even. The constraints (E10)-(E15) admit of 18 solutions. The first ten are

$$(x_{23}, x_{34}, y_{23}, y_{34}) = \left(0, \pm\frac{2\pi}{3}, \mp\frac{2\pi}{3}, \pm\frac{2\pi}{3}\right), \left(0, 0, \pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}\right), \left(0, \pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}, 0\right), \left(\pm\frac{2\pi}{3}, 0, \pm\frac{2\pi}{3}, \mp\frac{2\pi}{3}\right), \left(\pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{2\pi}{3}\right). \tag{E17}$$

The next eight solutions are identical to the first eight, except that each $x_{\ell m}$ is swapped with the corresponding $y_{\ell m}$.

Second, $n^{(\mathrm{i})} + n^{(\mathrm{iii})}$ can be even while $n^{(\mathrm{i})} + n^{(\mathrm{ii})}$ and $n^{(\mathrm{ii})} + n^{(\mathrm{iii})}$ are odd. The constraints (E10)-(E15) admit of another 18 solutions. The first ten are

$$(x_{23}, x_{34}, y_{23}, y_{34}) = \left(\pi, \pm\frac{\pi}{3}, \mp\frac{\pi}{3}, \pm\frac{\pi}{3}\right), \left(\pi, \pi, \pm\frac{\pi}{3}, \pm\frac{\pi}{3}\right), \left(\pi, \pm\frac{\pi}{3}, \pm\frac{\pi}{3}, \pi\right), \left(\pm\frac{\pi}{3}, \pi, \pm\frac{\pi}{3}, \mp\frac{\pi}{3}\right), \left(\pm\frac{\pi}{3}, \pm\frac{\pi}{3}, \pm\frac{\pi}{3}, \pm\frac{\pi}{3}\right). \tag{E18}$$

The next eight solutions are identical to the first eight, except that each $x_{\ell m}$ is swapped with the corresponding $y_{\ell m}$.

Third, $n^{(\mathrm{i})} + n^{(\mathrm{ii})}$ can be even while $n^{(\mathrm{i})} + n^{(\mathrm{iii})}$ and $n^{(\mathrm{ii})} + n^{(\mathrm{iii})}$ are odd. The constraints (E10)-(E15) admit of another 18 solutions. The first ten are

$$(x_{23}, x_{34}, y_{23}, y_{34}) = \left(0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{\pi}{3}\right), \left(0, \pi, \pm\frac{2\pi}{3}, \mp\frac{\pi}{3}\right), \left(0, \mp\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pi\right), \left(\pm\frac{2\pi}{3}, \pi, \pm\frac{2\pi}{3}, \pm\frac{\pi}{3}\right), \left(\pm\frac{2\pi}{3}, \mp\frac{\pi}{3}, \pm\frac{2\pi}{3}, \mp\frac{\pi}{3}\right). \tag{E19}$$

The next eight solutions are identical to the first eight, except that each $x_{\ell m}$ is swapped with the corresponding $y_{\ell m}$.

Fourth, suppose that $n^{(\mathrm{ii})} + n^{(\mathrm{iii})}$ is even while $n^{(\mathrm{i})} + n^{(\mathrm{ii})}$ and $n^{(\mathrm{i})} + n^{(\mathrm{iii})}$ are odd. The constraints (E10)-(E15) admit of another 18 solutions. The first ten are

$$(x_{23}, x_{34}, y_{23}, y_{34}) = \left(\pi, \pm\frac{2\pi}{3}, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}\right), \left(\pi, 0, \pm\frac{\pi}{3}, \mp\frac{2\pi}{3}\right), \left(\pi, \mp\frac{2\pi}{3}, \pm\frac{\pi}{3}, 0\right), \left(\pm\frac{\pi}{3}, 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}\right), \left(\pm\frac{\pi}{3}, \mp\frac{2\pi}{3}, \pm\frac{\pi}{3}, \mp\frac{2\pi}{3}\right). \tag{E20}$$

The next eight solutions are identical to the first eight, except that each $x_{\ell m}$ is swapped with the corresponding $y_{\ell m}$.

One can check explicitly that the tuple $(x_{23} + y_{23},\ x_{34} + y_{34})$ has three possible values: $(x_{23} + y_{23},\ x_{34} + y_{34}) = (\pm 2\pi/3,\ \pm 2\pi/3), (\pm 4\pi/3,\ \pm 4\pi/3), (\pm 2\pi/3,\ \mp 4\pi/3)$. Three sets of solutions follow. For example, the first set of solutions is $(x_{23} + y_{23},\ x_{34} + y_{34}) = (\pm 2\pi/3,\ \pm 2\pi/3)$. Hence

$$a^{(\mathrm{i})} - a^{(\mathrm{ii})} + b^{(\mathrm{i})} - b^{(\mathrm{ii})} = \pm\frac{2\pi}{3}, \qquad a^{(\mathrm{ii})} - a^{(\mathrm{iii})} + b^{(\mathrm{ii})} - b^{(\mathrm{iii})} = \pm\frac{2\pi}{3}, \tag{E21}$$

$$a^{(\ell)} - a^{(m)} \in \left\{0, \frac{\pm\pi}{3}, \frac{\pm 2\pi}{3}, \pi\right\}, \quad \text{and} \quad b^{(\ell)} - b^{(m)} \in \left\{0, \frac{\pm\pi}{3}, \frac{\pm 2\pi}{3}, \pi\right\}, \tag{E22}$$

for $(\ell, m) = (2,3)$ and $(3,4)$. All the solutions lead to the same Hamiltonian, Eq. (25).

E 3 Ladder operators for $\mathfrak{su}(3)$

The conventional Cartan-Weyl basis contains six ladder operators [Eqs. (24)]. We transform $L_{\pm 1,2,3}$ with the unitaries $U_{\rm i}$, $U_{\rm ii}$, and $U_{\rm iii}$ of App. E 2, to construct the rest of the ladder operators: $L_{\pm 4} = U_{\rm i}^\dagger L_{\pm 1} U_{\rm i}$, $L_{\pm 5} = U_{\rm i}^\dagger L_{\pm 2} U_{\rm i}$, and $L_{\pm 6} = U_{\rm i}^\dagger L_{\pm 3} U_{\rm i}$. Substituting in for $L_{\pm 1,2,3}$ from Eq. (24) yields

$$
\begin{aligned}
L_{\pm 4} = \frac{i e^{\mp i \phi_1^{(\rm i)}}}{6} \Big\{ & 2i \cos\Big(a^{(\rm i)} - b^{(\rm i)}\Big) \lambda_1 - 2i \sin\Big(a^{(\rm i)} - b^{(\rm i)}\Big) \lambda_2 \mp \Big[\sqrt{3} \mp i(-1)^{n^{(\rm i)}}\Big] \Big[\cos\Big(a^{(\rm i)}\Big)\lambda_4 - \sin\Big(a^{(\rm i)}\Big)\lambda_5\Big] \\
& \pm \Big[\sqrt{3} \pm i(-1)^{n^{(\rm i)}}\Big] \Big[\cos\Big(b^{(\rm i)}\Big)\lambda_6 - \sin\Big(b^{(\rm i)}\Big)\lambda_7\Big] \mp \sqrt{3}(-1)^{n^{(\rm i)}}\lambda_3 - \sqrt{3} i \lambda_8 \Big\},
\end{aligned} \tag{E23}
$$

$$
\begin{aligned}
L_{\pm 5} = \frac{i e^{\mp \frac{i}{2}(\phi_3^{(\rm i)} + \phi_1^{(\rm i)})}}{6} \Big(& i \Big[\cos\Big(a^{(\rm i)} - b^{(\rm i)}\Big) - \sqrt{3}(-1)^{n^{(\rm i)}} \sin\Big(a^{(\rm i)} - b^{(\rm i)}\Big)\Big] \lambda_1 \\
& - i \Big[\sin\Big(a^{(\rm i)} - b^{(\rm i)}\Big) + \sqrt{3}(-1)^{n^{(\rm i)}} \cos\Big(a^{(\rm i)} - b^{(\rm i)}\Big)\Big] \lambda_2 \pm \frac{1}{2} \Big\{(-1)^{n^{(\rm i)}} \Big[3 \sin\Big(a^{(\rm i)}\Big) \pm i \cos\Big(a^{(\rm i)}\Big)\Big] + \sqrt{3} e^{\pm i a^{(\rm i)}}\Big\} \lambda_4 \\
& \pm \frac{1}{2} \Big\{(-1)^{n^{(\rm i)}} \Big[3 \cos\Big(a^{(\rm i)}\Big) \mp i \sin\Big(a^{(\rm i)}\Big)\Big] \pm i\sqrt{3} e^{\pm i a^{(\rm i)}}\Big\} \lambda_5 \\
& \pm \frac{1}{2} \Big\{(-1)^{n^{(\rm i)}} \Big[3 \sin\Big(b^{(\rm i)}\Big) \pm i \cos\Big(b^{(\rm i)}\Big)\Big] - \sqrt{3} e^{\pm i b^{(\rm i)}}\Big\} \lambda_6 \\
& \pm \frac{1}{2} \Big\{(-1)^{n^{(\rm i)}} \Big[3 \cos\Big(b^{(\rm i)}\Big) \mp i \sin\Big(b^{(\rm i)}\Big)\Big] \mp i\sqrt{3} e^{\pm i b^{(\rm i)}}\Big\} \lambda_7 \mp \sqrt{3}(-1)^{n^{(\rm i)}} \lambda_3 + \sqrt{3} i \lambda_8 \Big), \quad \text{and}
\end{aligned} \tag{E24}
$$

$$
\begin{aligned}
L_{\pm 6} = \frac{i e^{\mp \frac{i}{2}\left(\phi_3^{(\rm i)} - \phi_1^{(\rm i)}\right)}}{6} \Big(& - i \Big[\cos\Big(a^{(\rm i)} - b^{(\rm i)}\Big) + \sqrt{3}(-1)^{n^{(\rm i)}} \sin\Big(a^{(\rm i)} - b^{(\rm i)}\Big)\Big] \lambda_1 \\
& + i \Big[\sin\Big(a^{(\rm i)} - b^{(\rm i)}\Big) - \sqrt{3}(-1)^{n^{(\rm i)}} \cos\Big(a^{(\rm i)} - b^{(\rm i)}\Big)\Big] \lambda_2 \mp \frac{1}{2} \Big\{(-1)^{n^{(\rm i)}} \Big[3 \sin\Big(a^{(\rm i)}\Big) \pm i \cos\Big(a^{(\rm i)}\Big)\Big] - \sqrt{3} e^{i a^{(\rm i)}}\Big\} \lambda_4 \\
& \mp \frac{1}{2} \Big\{(-1)^{n^{(\rm i)}} \Big[3 \cos\Big(a^{(\rm i)}\Big) \mp i \sin\Big(a^{(\rm i)}\Big)\Big] \mp i\sqrt{3} e^{i a^{(\rm i)}}\Big\} \lambda_5 \mp \frac{1}{2} \Big\{(-1)^{n^{(\rm i)}} \Big[3 \sin\Big(b^{(\rm i)}\Big) \pm i \cos\Big(b^{(\rm i)}\Big)\Big] + \sqrt{3} e^{i b^{(\rm i)}}\Big\} \lambda_6 \\
& \mp \frac{1}{2} \Big\{(-1)^{n^{(\rm i)}} \Big[3 \cos\Big(b^{(\rm i)}\Big) \mp i \sin\Big(b^{(\rm i)}\Big)\Big] \pm i\sqrt{3} e^{i b^{(\rm i)}}\Big\} \lambda_7 \mp \sqrt{3}(-1)^{n^{(\rm i)}} \lambda_3 - \sqrt{3} i \lambda_8 \Big).
\end{aligned} \tag{E25}
$$

$L_{\pm 7}$, $L_{\pm 8}$, and $L_{\pm 9}$ have the same forms. However, (ii)'s replace the superscripts (i)'s. $L_{\pm 10}$, $L_{\pm 11}$, and $L_{\pm 12}$ likewise have the same form, except that (iii)'s replace the (i)'s.

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