Noncommuting charges' effect on the thermalization of local observables

Shayan Majidy[1, 2, *]

[1] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*

[2] *Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

(Dated: March 21, 2024)

Studying noncommuting conserved quantities (or 'charges') has produced a conceptual puzzle. Recent results suggest that noncommuting charges hinder thermalization in some ways, yet promote it in others. To help resolve this puzzle, we demonstrate how noncommuting charges can promote thermalization by reducing the number of local observables that thermalize according to the Eigenstate Thermalization Hypothesis. We first establish a correspondence between charges and sufficient conditions for observables to not thermalize. These conditions are known as 'dynamical symmetries.' We demonstrate that for each pair of dynamical symmetries a Hamiltonian has, there exists a corresponding charge. We prove that the reciprocal relationship holds for a broad range of charges. From this correspondence, we demonstrate that introducing a new charge to a system can either contribute to or disrupt its existing non-stationary dynamics. If the new charge commutes with existing ones, the system's non-stationary dynamics remain intact, and new ones emerge; if not, the existing non-stationary dynamics are removed. We illustrate our results using variou models. Our results demonstrate a facet of thermalization which noncommuting charges promote.

I. INTRODUCTION

How do isolated quantum many-body systems come to thermal equilibrium? This question is largely answered by the eigenstate thermalization hypothesis (ETH) [1, 2]. Consider a set of local observables O_i, an initial pure state[1] $|\psi\rangle$ governed by a Hamiltonian H, and the corresponding expectation values $\langle O_i(t)\rangle$. We say a system is in thermal equilibrium at time t if $\langle O_i(t)\rangle \approx \langle O_i\rangle_{\text{th}} := \text{tr}[\rho_{\text{th}} O_i]$ for each O_i, where ρ_{th} is the thermal state with temperature fixed by the energy of the initial state. If the Hamiltonian and local observables satisfy the ETH, the system will thermalize in this sense. Despite the apparent tension between unitary dynamics and thermalization, most quantum many-body systems thermalize [3].

A puzzle related to thermalization has emerged from the study of noncommuting charges [4–7]. Traditionally, it was assumed that conserved quantities, or 'charges', would commute, a premise underlying derivations of the thermal state's form [5–8], Onsager coefficients [9], and ETH [10]. However, in quantum theory, the noncommutation of observables is central, playing a key role in uncertainty relations [11, 12], measurement disturbance [13, 14], and tests of quantum theory [15–18]. Therefor, removing this assumption is vital and has catalyzed new results in quantum thermodynamics and many-body physics. Noncommuting charges can, for example, increase average entanglement [19], invoke critical dynamics [20], and decrease the rate of entropy production [21, 22]. Additionally, noncommuting charges impose stricter constraints on the implementable unitaries than commuting charges do [23] and, based on a phyically plausible assumption, lead to larger deviations from the thermal state [10].

The results above present an interesting puzzle: noncommuting charges appear to both hinder and promote thermalization in different contexts [4, 19]. This dichotomy is theoretically intriguing and may have implications for quantum technologies. The main challenge in developing scalable quantum computers is mitigating decoherence [24–26], with thermalization being a significant cause. If systems with noncommuting charges resist thermalization, they could contribute to more decoherence-resistant quantum technologies. Systems with noncommuting charges, such as spin systems [10, 19, 27, 28] and squeezed states [9, 22], are naturally present in quantum computing methods, like quantum dot [29] and optical approaches [30]. This potential is further supported by recent advances showing that non-Abelian symmetric operations are universal for quantum computing [31, 32].

To help resolve this puzzle, we aim to connect noncommuting charges and conditions that lead to violations of the ETH. Buča et al. [33] proposed a set of conditions wherein if an operator A satisfies $[H, A] = \lambda A$ with $\lambda \in \mathbb{R}$ and $\lambda \neq 0$, then any local operator $\mathcal{O}_i$ overlapping with A (i.e., $\text{tr}[\mathcal{O}_i A] \neq 0$) will not thermalize, displaying non-stationary dynamics in violation of the ETH [33]. These operators, A, are referred to as dynamical symmetries' [33–36] or spectrum generating algebras' [37].[2] Dynamical symmetries represent a departure from the conventional definition of 'symmetry' as an invariance under a transformation. Therefore, the link between dynamical symmetries and charges is not directly established by Noether's theorem. These symmetries prevent a quantum many-body system from reaching a stationary state,

* smajidy@uwaterloo.ca

[1] Having a pure initial state is not necessary for the ETH. However, the paradox of unitary dynamics leading to thermalization is most pronounced in pure states; therefore, we consider them.

[2] The terms 'dynamical symmetry' and 'spectrum generating algebra' encompass a range of related concepts [38, 39]. In this paper, we specifically refer to the condition previously stated.

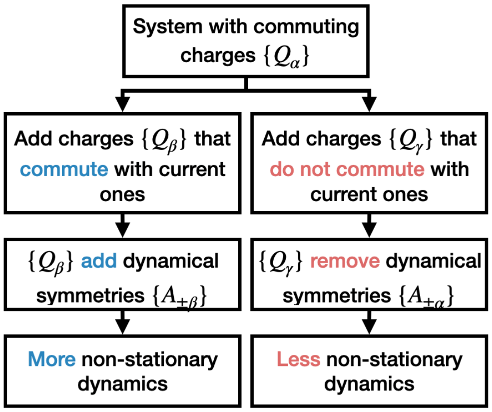

Figure 1: Introducing charges into systems with existing dynamical symmetries. A system with commuting charges $\{Q_\alpha\}$ possesses paired dynamical symmetries $\{A_{\pm\alpha}\}$. Altering the system's Hamiltonian to conserve non-commuting charges $\{Q_\beta\}$ relative to $\{Q_\alpha\}$ will result in the loss of some or all dynamical symmetries $\{A_{\pm\alpha}\}$. Conversely, a different modification that introduces new commuting charges will bring associated dynamical symmetries. These new charges have no algebraic relationship with the initial charges, suggesting an increase in the system's dynamical symmetries.

affecting both open [33, 40, 41] and closed [34] quantum systems. Dynamical symmetries can be responsible for the non-stationary dynamics seen in systems such as quantum time crystals [34], OTO crystals [42], and quantum attractors [43].

Our primary finding is the correlation between charges and dynamical symmetries. We establish that for each pair of dynamical symmetries, there exists a corresponding charge, and we detail the method for deriving this charge from the dynamical symmetries. Additionally, we confirm that this reciprocal relationship extends to a wide range of charges. This correlation illustrates that even a single charge can cause a violation of the ETH. Through this framework, we show that introducing a new charge to a system can either enhance or disrupt its non-stationary dynamics. Specifically, if the new charge commutes with the existing ones, it will preserve the current dynamical symmetries and introduce new ones. Conversely, if the new charge does not commute, it will reduce the number of dynamical symmetries. Figure 1 encapsulates these findings. Our analyses reveal a fundamental tension between noncommuting charges and dynamical symmetries, and highlights that the introduction of additional conservation laws can drive a system towards thermalization.

Dynamical symmetries are also linked to quantum scars, a class of eigenstates that do not thermalize [44–47]. Ref.[48] describes creating Hamiltonians with quantum scars by altering a non-Abelian symmetric Hamiltonian with a dynamical symmetry-based term. Our research contrasts this by examining how noncommuting and commuting charges affect various systems' dynamical symmetries and establishing a related correspondence. Unlike the quantum scar focus, our analysis targets local observables and can naturally extend beyond Hamiltonians to Lindbladians, since noncommuting charges and dynamical symmetries are relevant in both closed and open systems [33, 40, 41, 49]. Future connections are explored in the Outlook section (Sec. V).

The manuscript is structured as follows: Section II reviews the ETH, dynamical symmetries, and noncommuting charges. In Section III, we introduce our first main result, detailing the correspondence between dynamical symmetries and charges. Section IV discusses our second result, which shows how introducing a new charge to a system affects its non-stationary dynamics, contingent on the commutation with existing charges, illustrated through various examples. The conclusion in Section V connects our findings with existing research and outlines directions for future work.

II. BACKGROUND

Consider a closed quantum system consisting of a lattice with N sites. Each site corresponds to a Hilbert space $\mathcal{H}$ of finite dimensionality d. The system is governed by a Hamiltonian $H = \sum_k E_k |\psi_k\rangle\langle\psi_k|$, where $|\psi_k\rangle$ are energy eigenstates with energies E_k. The time dependent state $|\Phi(t)\rangle = \sum_k \exp(-iE_k t)c_k |\psi_k\rangle$ will have fixed total energy $E = \langle\Phi(t)|H|\Phi(t)\rangle$, where we set $\hbar = 1$. The expectation value of an observable $\mathcal{O}$ for the state $|\Phi(t)\rangle$ is

$$\langle\mathcal{O}(t)\rangle = \sum_{j,k} e^{-i(E_k-E_j)t} c_j^* c_k \langle\psi_j|\mathcal{O}|\psi_k\rangle . \tag{1}$$

$\langle\mathcal{O}\rangle_{\text{th}} := \text{tr}[\rho_{\text{th}} O]$ is the thermal expectation value where ρ_{th} is the thermal state. If an out-of-equilibrium $|\Phi(0)\rangle$ thermalizes, we expect $\langle\mathcal{O}(t)\rangle$ to approach $\langle\mathcal{O}\rangle_{\text{th}}$,

$$\langle\mathcal{O}(t\to\infty)\rangle = \langle\mathcal{O}\rangle_{\text{th}} + C, \tag{2}$$

where C accounts for fluctuations. The ETH gives a set of conditions for which this expectation holds.

Systems with dynamical symmetries violate the ETH [33, 35]. Dynamical symmetries can, for example, lead Heisenberg XXZ spin-chains to behave as quantum time crystals [34] and spin lace systems to behave as quantum many-body attractors [43]. We denote a dynamical symmetry by a non-Hermitian operator A, such that

$$[H, A] = \lambda A, \tag{3}$$

where λ is a real and non-zero constant. We assume that A is extensive, by restricting it to have the form $A = \sum_{j=1}^N \tilde{A}^{(j)}$, where $\tilde{A}^{(j)}$ is an operator that acts nontrivially on 1 site and as the identity on all other sites. Furthermore, for every dynamical symmetry A, there exists another $A^\dagger$, $\left[H, A^\dagger\right] = -\lambda A^\dagger$. Thus, dynamical symmetries always come in pairs. A system can have multiple pairs of dynamical symmetries. In that case, we add subscripts as follows $[H, A_\beta] = \lambda_\beta A_\beta$. Furthermore, we define $A_{+\beta} := A_\beta$ and $A_{-\beta} := A_\beta^\dagger$. The expectation value of an observable $\mathcal{O}$ that overlaps with one dynamical symmetry, $\mathrm{tr}[A\mathcal{O}] \neq 0$, will continue to change in time.

Finally, we introduce noncommuting charges. Charges are Hermitian operators that commute with the Hamiltonian, $[H, Q] = 0$. A system can have multiple charges, that we distinguish with Greek letter subscripts, and these charges can be noncommuting, $[Q_\alpha, Q_\beta] \neq 0$. For physically motivated reasons, we add a third condition to hermiticity and commutation. Let $\tilde{Q}_\alpha$ denote a Hermitian operator defined on $\mathcal{H}$. We denote by $\tilde{Q}_\alpha^{(j)}$ the $\tilde{Q}_\alpha$ defined on the j^{th} subsystem's $\mathcal{H}$. We denote an extensive observable

$$Q_\alpha := \sum_{j=1}^{N} \tilde{Q}_\alpha^{(j)} \equiv \sum_{j=1}^{N} \mathbb{1}^{\otimes(j-1)} \otimes \tilde{Q}_\alpha^{(j)} \otimes \mathbb{1}^{\otimes(N-j)}. \quad (4)$$

Our third condition is that the charges are extensive. While this work focuses on closed quantum systems, noncommuting charges and dynamical symmetries can also exist in open quantum systems [33, 40, 41, 49].

III. CORRESPONDENCE BETWEEN CHARGES AND DYNAMICAL SYMMETRIES

In this section, we present a correspondence between the existence of charges and dynamical symmetries. This correspondence is in the form of two theorems. One theorem identifies a charge from pairs of dynamical symmetries. The other theorem identifies a pair of dynamical symmetries from charges. We first prove this correspondence (Sec. III A) before illustrating it using the Hubbard model (Sec. III B).

III A. Correspondence

Charges Q_α that generate Lie algebras are important in physics because they describe, for example, angular momentum, particle number, electric charge, color charge, and weak isospin [4, 50–52], i.e., everything in the Standard Model of particle physics. The Lie algebras relevant to studying noncommuting charges are finite-dimensional because we study systems with a finite number of linearly independent charges [28]. The algebras are also defined over the complex number because the operators are Hermitian. Finally, the algebras are semisimple so that the operator representation of the charges can be diagonalized (not necessarily simultaneously diagonalized) [53]. Many physically significant algebras satisfy these conditions, such as $\mathfrak{su}(N)$, $\mathfrak{so}(N)$, and all simple Lie algebras. From this point onward, we denote by $\mathfrak{g}$ a finite-dimensional semisimple complex Lie algebra.

The results of this paper apply to all of the charges mentioned above and others, which we will now introduce. An algebra's dimension c equals the number of generators in a basis for the algebra. The algebra's rank r is the dimension of the algebra's maximal commuting subalgebra, the largest subalgebra in which all of the elements are commuting. For example, consider the usual basis for $\mathfrak{su}(2)$—the Pauli-operators $\{\sigma_x, \sigma_y, \sigma_z\}$. There are three generators in this basis, so $c = 3$, and none of these operators commute with one another, so $r = 1$. A Cartan subalgebra is a maximal Abelian subalgebra consisting of semisimple elements, $h \in \mathfrak{h}$. From this point onward, we denote by $\mathfrak{h}$ a Cartan subalgebra of a $\mathfrak{g}$. Every $\mathfrak{g}$ will have a $\mathfrak{h}$. Our results apply to all sets of charges $\{Q_\alpha\}$ that can be partitioned into subsets such that each subset generates a $\mathfrak{g}$ or a $\mathfrak{h}$. Note that this includes all sets $\{Q_\alpha\}$ that generate a $\mathfrak{g}$ and all sets $\{Q_\alpha\}$ that generate a $\mathfrak{h}$, even if they may not generate a full $\mathfrak{g}$. This is a wide class that includes, for example, everything in the Standard Model of physics and the charges of the Hubbard, Ising, and Heisenberg models.

Essential to our study are Cartan-Weyl bases [53]. The $\mathfrak{g}$ definition includes a vector space V defined over a field F. A form is a map $V \times V \to F$. The Killing form of operators $x, y \in \mathfrak{g}$ is the bilinear form $(x, y) := \mathrm{tr}(\mathrm{ad}x \cdot \mathrm{ad}y)$ where $\mathrm{ad}x$ is the image of x under the adjoint representation of $\mathfrak{g}$. Let $\beta(h) := (h'_\beta, h)$ where $h, h'_\beta \in \mathfrak{h}$. $\beta(h)$ is a root of $\mathfrak{g}$ relative to $\mathfrak{h}$ if there exists a non-zero operator $X_\beta \in \mathfrak{g}$ such that

$$[h, X_\beta] = \beta(h) X_\beta. \quad (5)$$

These operators X_β are called root vectors. Denote by Δ all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. If $\beta \in \Delta$, then so is $-\beta$. Thus, root vectors always come in pairs $X_{\pm\beta}$. Finally, one can identify different Cartan-Weyl bases for each $\mathfrak{g}$. A Cartan-Weyl basis consists of a Cartan subalgebra $\mathfrak{h}$ and one or more pairs of root vectors $X_{\pm\beta}$. The choice of Cartan-Weyl is not unique.

We highlight additional features of root vectors which are important to this work. For all root vectors $[X_{+\beta}, X_{-\beta}] \in \mathfrak{h}$ (Proposition 7.17 of Ref. [54]). $X_{\pm\beta}$ can always be chosen such that $X_{+\beta} = X_{-\beta}^\dagger$ (p. 273 of Ref. [55]). It follows then that

$$([X_\beta, X_{-\beta}])^\dagger = [X_\beta, X_{-\beta}] \quad (6)$$

and that

$$[X_{+\beta}, [\mathcal{H}, X_{-\beta}]] = ([X_{-\beta}, [\mathcal{H}, X_{+\beta}]])^\dagger \quad (7)$$

where $\mathcal{H}$ is any Hermitian operator.

Theorem 1. *For every pair of dynamical symmetries $A_{\pm\beta}$ that a Hamiltonian has, there exists a charge $Q_\beta = [A_{+\beta}, A_{-\beta}]$.*

Proof. For Q_β to be a charge, it must be conserved by the Hamiltonian, a Hermitian operator, and extensive.

Using the Jacobi identity, Eq. (6), and Eq. (7), we find that

$$[H,[A_{+\beta},A_{-\beta}]] = [A_{+\beta},[H,A_{-\beta}]] - [A_{-\beta},[H,A_{+\beta}]] \tag{8}$$
$$= -\lambda_\beta\left([A_{+\beta},A_{-\beta}] + [A_{-\beta},A_{+\beta}]\right) \tag{9}$$
$$= 0. \tag{10}$$

Thus, the Hamiltonian conserves Q_β. For Q_β to be a charge, it must also be Hermitian, which it is

$$Q_\beta^\dagger = \left(A_\beta A_\beta^\dagger - A_\beta^\dagger A_\beta\right)^\dagger \tag{11}$$
$$= \left(A_\beta A_\beta^\dagger - A_\beta^\dagger A_\beta\right) = Q_\beta. \tag{12}$$

Finally, recall that we are studying dynamical symmetries, which are $k=1$ local. Thus, all $Q_\beta = [A_{+\beta}, A_{-\beta}]$ will also be 1-local and are charges by our definition. □

We say a set of dynamical symmetries 'produces' a set of charges $\{Q_\beta\}$ when the span of $\{Q_\beta\}$ equals the span of the set of charges found using Theorem 1.

Theorem 2. *For every set of charges $\{Q_\beta\}$ that form a $\mathfrak{h}$ of $\mathfrak{g}$, the root vectors that complete the Lie algebra are dynamical symmetries of H.*

Proof. Since the charges Q_β form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $[H,h]=0$ for all $h\in\mathfrak{h}$. Thus, $H\in\mathfrak{h}$ since $\mathfrak{h}$ is the maximal Abelian subalgebra. Recall that the commutator of an element of the Cartan subalgebra with a root vector equals the corresponding root times the root vector, $H\in\mathfrak{h}$, $[H,X_{+\beta}] = \beta(H)X_{+\beta}$. This immediately has the form of a dynamical symmetry. Thus, all root vectors of $\mathfrak{g}$ are dynamically symmetries of H. □

We say a set of charges 'produces' a set of dynamical symmetries $\{A_{\pm\beta}\}$, when $\{A_{\pm\beta}\}$ equals one set of dynamical symmetries that can be found using Theorem 2.

Theorem 2 is for a set of charges that form a $\mathfrak{h}$ of $\mathfrak{g}$. Any set of charges that generate a $\mathfrak{g}$ can be partitioned into $\frac{c}{r}$ sets of mutually commuting charges [28]. Thus, through the repeated application of Thm. 2, this theorem can be used to study charges that generate $\mathfrak{g}$.

III B. Illustration using the Hubbard model

A simple setting to illustrate this correspondence is the Hubbard model. We choose this model for various reasons. First, it has been shown to demonstrate nonstationary behaviour emerging from dynamical symmetries and the forms of the charges and dynamical symmetries are known [33]. Additionally, its two commuting charges form separate $\mathfrak{h}$'s, illustrating the effect commuting charges have on dynamical symmetries. Finally, it is a physically important model—the prototypical model of strongly correlated materials.

Consider a chain of N fermions. Denote by $c_\sigma^{(j)\dagger}$ and $c_\sigma^{(j)}$ the creation and annihilation operators for a fermion of spin σ at lattice site j, $\sigma\in\{\uparrow,\downarrow\}$. Denote by $n_\sigma^{(j)} := c_\sigma^{(j)\dagger}c_\sigma^{(j)}$ the number operator for fermions of spin σ at lattice site j. The 1D Hubbard model's Hamiltonian can be written as

$$H = \sum_{j=1}^{N-1}\sum_{\sigma=\uparrow,\downarrow} -t\left(c_\sigma^{(j)\dagger}c_\sigma^{(j+1)} + c_\sigma^{(j+1)\dagger}c_\sigma^{(j)}\right) + Un_\uparrow^{(j)}n_\downarrow^{(j)} - \mu\left(n_\uparrow^{(j)} + n_\downarrow^{(j)}\right) + \frac{B}{2}\left(n_\uparrow^{(j)} - n_\downarrow^{(j)}\right), \tag{13}$$

where t is the hopping parameter, U is the on-site Coulomb interaction, μ is the chemical potential, and B is the strength of a constant external magnetic field.

The Hubbard model has two pairs of dynamical symmetries [33]. The first pair are

$$S_{+z}^{\text{tot}} = \sum_{j=1}^{L} c_\uparrow^{(j)\dagger}c_\downarrow^{(j)} \quad\text{and}\quad S_{-z}^{\text{tot}} = \sum_{j=1}^{L} c_\downarrow^{(j)\dagger}c_\uparrow^{(j)}, \tag{14}$$

and the second pair are

$$\eta_{+z}^{\text{tot}} = \sum_{j=1}^{N}(-1)^j c_\uparrow^{(j)\dagger}c_\downarrow^{(j)\dagger} \text{ and } \eta_{-z}^{\text{tot}} = \sum_{j=1}^{L}(-1)^j c_\downarrow^{(j)}c_\uparrow^{(j)}. \tag{15}$$

Using theorem 1, we identify two charges:

$$\left[S_{+z}^{\text{tot}}, S_{-z}^{\text{tot}}\right] = S_z^{\text{tot}} = \sum_{j=1}^{L}\left(n_\uparrow^{(j)} - n_\downarrow^{(j)}\right), \text{ and} \tag{16}$$

$$\left[\eta_{+z}^{\text{tot}}, \eta_{-z}^{\text{tot}}\right] = \eta_z^{\text{tot}} = \sum_{j=1}^{L}(n_\uparrow^{(j)} + n_\downarrow^{(j)} - 1). \tag{17}$$

These charges are the two known charges of the system [56].

Starting from the charges, we can also identify the dynamical symmetries using theorem 2. When $B=0$ and $\mu=0$, the Hubbard Hamiltonian contains two sets of charges that generate $\mathfrak{su}(2)$ [57]. For $B\neq0$ and $\mu\neq0$, the Hubbard model has two sets of charges that generate Cartan subalgebras of $\mathfrak{su}(2)$. Each charge, S_z^{tot} and η_z^{tot}, is an element in one of these algebras. We can use these Cartan subalgebras to complete a Cartan-Weyl basis for $\mathfrak{su}(2)$. Doing so, we find two sets of generators $\{S_z^{\text{tot}}, S_{+z}^{\text{tot}}, S_{-z}^{\text{tot}}\}$ and $\{\eta_z^{\text{tot}}, \eta_{+z}^{\text{tot}}, \eta_{-z}^{\text{tot}}\}$. This demonstrates how Cartan-Weyl bases can be used to identify the dynamical symmetries from the charges.

IV. NONCOMMUTING CHARGES' EFFECT ON DYNAMICAL SYMMETRIES

In this section, we consider the following setting. We begin with a system that has a set of charges $\{Q_\alpha\}$ that, according to theorem 2, produce dynamical symmetries $\{A_{\pm\alpha'}\}$. This system experiences non-stationary dynamics in all observables $\mathcal{O}$ that overlap with any elements in $\{A_{\pm\alpha'}\}$. We then introduce one or more charges into the system. This introduction of charge(s) can either add to or disrupt its existing non-stationary dynamics. A summary of this section's result are presented in Fig. 1. First, to be mathematically rigorous, we formalize our procedure for introducing charges in Sec. IV A. Our procedure highlights why there is a difference in the commuting and noncommuting charge cases. In Sec. IV B, we illustrate this procedure using charges that generates a Cartan subalgebra of $\mathfrak{su}(2)$ and to charges that generate $\mathfrak{su}(2)$. We do the same analysis in Sec. IV C for the $\mathfrak{su}(3)$ algebra. These sections further demonstrate the difference in introducing commuting and noncommuting charges. We conclude by presenting Hamiltonians in Section IV D that can be used explore the distinction between commuting and noncommuting charges. This connection aims to link our findings with experimental tests and related research on quantum scars.

IV A. Procedure

Consider a system with a set of charges $\{Q_\alpha\}$ that can be partitioned into subsets such that each subset generates a $\mathfrak{g}$ or a $\mathfrak{h}$. In the earlier Hubbard model example, we had two subsets that each generate a $\mathfrak{h}$ (S_z^{tot} and η_z^{tot}). We want a procedure to identify a system's dynamical symmetries from these charges. Below is a procedure to do so for one of the subsets. The subset of charges generates a $\mathfrak{h}$ in 'scenario 1' and a $\mathfrak{g}$ in 'scenario 2'.

1. Partition the system's c charges into $\frac{c}{r}$ sets of mutually commuting charges. In scenario 1, this will be all the charges. In scenario 2, such a partition always exists and generates a $\mathfrak{h}$ [28].

2. Select any one of $\mathfrak{h}$'s identified in step 1.

3. Construct a Cartan-Weyl basis by adding to $\mathfrak{h}$ $\frac{c-r}{2}$ pairs of root vectors, $X_{\pm\beta}$.

4. According to Theorem 2, these pairs of root vectors are dynamical symmetries. We add them to our list of the system's dynamical symmetries, $A_{\pm\beta} := X_{\pm\beta}$. The values of λ_β can be determined in two ways. One is that they are equal to the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ for a given charge $Q'_\alpha \in \mathfrak{h}'$. Thus, one can solve for $\beta(h)$ and set λ_β equal to that. Alternatively, one can explicitly solve for $[H, A_\beta] = \lambda_\beta A_\beta$. The specific $\mathfrak{g}$ will determine which is simpler to do.

5. For scenario 1, the procedure ends here. For scenario 2, select a different one of the $\frac{c}{r}$ Cartan subalgebras identified in step 1. Verify whether including the new charges removes any of the dynamical symmetries found earlier.

6. Repeat steps 3 to 5 until no further Cartan subalgebras remain.

One can then repeat this procedure for each of the subsets mentioned above. For each set of charges, this procedure identifies $\frac{c}{r}(c-r)$ dynamical symmetries of the system from c charges.

The procedure can naturally be reversed using Theorem 1, where one identifies a charge from each pair of dynamical symmetries. The reverse procedure uses $\frac{c}{r}(c-r)$ dynamical symmetries to identify $\frac{c}{2r}(c-r)$ charges. This is because the charges identified in the reverse procedure are generally not linearly independent. The linear independence of charges Q_α generating a Lie algebra $\mathfrak{g}$ can be assessed by computing the Killing forms between all pairs of Q_α. These charges are linearly independent if all the Killing forms are 0 [28]. However, from a linearly dependent set of charges, one can easily form a linearly independent set by summing over different charges.

In our framework, charges can commute in two ways. First, they may be components of distinct algebras, as exemplified in the Hubbard model. Alternatively, they can belong to the same Cartan subalgebra, $\mathfrak{h}$, which will be illustrated with the $\mathfrak{su}(3)$ example. In this scenario, the procedure adds dynamical symmetries for the charges at Step 4. It is important to remember that root vectors are associated with specific charges, as indicated in Eq. (5). For clarity, we can deconstruct this step by sequentially integrating the dynamical symmetries for each charge within $\mathfrak{h}$. This approach elucidates that introducing a new charge preserves the dynamical symmetries established by the preceding charges.

Charges noncommutation enters the picture in step 5. If the charges generate a $\mathfrak{g}$, there is the possibility that the full set of charges does not commute. The Cartan-Weyl basis is a basis for $\mathfrak{g}$. The charges also form a basis for $\mathfrak{g}$. Thus, the elements of the Cartan-Weyl basis can be written in the basis of the charges, i.e., the dynamical symmetries can be written as a linear combination of charges. Thus, if the Hamiltonian commutes with more charges, it will commute with more dynamical symmetries. This mechanism explains how introducing noncommuting charges leads to the removal of existing dynamical symmetries.

IV B. Illustration using $\mathfrak{su}(2)$

First, we consider charges represented by $\mathfrak{su}(2)$. $\{\sigma_x, \sigma_y, \sigma_z\}$ are the usual Pauli operators. Consider again a system of N sites. We denote by $\sigma_\alpha^{(j)}$ a Pauli operator acting on the jth site. We define the operators $S_\alpha^{\text{tot}} := \sum_{j=1}^N \sigma_\alpha^{(j)}$. If we were studying the full algebra, we would have three charges that are the components of the spin-$\frac{1}{2}$ angular momentum.

First, we will consider having a single Cartan subalgebra and return to the full algebra later. The system has $c = 3$ and $r = 1$. Thus, its maximal Abelian subalgebra will have one element. To be concrete, we choose this element to be $S_z^{\rm tot}$. Next, we complete the Cartan-Weyl basis by identifying the root vectors of the algebra, which are

$$S_{\pm z}^{\rm tot} = \sum_{j=1}^{N} \mathbb{1}^{\otimes(j-1)} \otimes S_{\pm z}^{(j)} \otimes \mathbb{1}^{\otimes(N-j)} \equiv \sum_{j=1}^{N} S_{\pm z}^{(j)} \quad (18)$$

where $S_{\pm z} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. Starting from the system's charge, we identified two dynamical symmetries. Like with the Hubbard model, it is straightforward to reverse this procedure. Doing so, we check that $[S_{+z}, S_{-z}] = S_z$, and thus $\left[S_{+z}^{\rm tot}, S_{-z}^{\rm tot}\right] = S_z^{\rm tot}$.

The system described above has one charge and one pair of dynamical symmetries. We now want to introduce another charge that does not commute with the existing one, such as $S_x^{\rm tot}$. However, a Hamiltonian that conserves two charges of $\mathfrak{su}(2)$ will necessarily conserve all three [28]. Thus, we introduce two more charges into the system, thereby applying the the procedure from Sec. IV A for $\mathfrak{su}(2)$. The first round of steps 1 to 4 for finding the dynamical symmetries of the full $\mathfrak{su}(2)$ is equivalent to finding the dynamical symmetries of one Cartan subalgebra of $\mathfrak{su}(2)$. To complete $\mathfrak{su}(2)$, we include $S_x^{\rm tot}$ and $S_y^{\rm tot}$ as charges. Say we did not check whether these new charges remove earlier dynamical symmetries in step 5. We would then choose one of these two charges and repeat steps 1 to 4. Doing so we would find the dynamical symmetries $S_{\pm\alpha}^{\rm tot} = \sum_{j=1}^{N} S_{\pm\alpha}^{(j)}$ for $\alpha = x$ and y, where $S_{\pm x} = \frac{1}{2}(\sigma_z \mp i\sigma_y)$ and $S_{\pm y} = \frac{1}{2}(\sigma_z \pm i\sigma_x)$. It is straightforward to check that $\left[S_{+x}^{\rm tot}, S_{-x}^{\rm tot}\right] = S_x^{\rm tot}$ and $\left[S_{+y}^{\rm tot}, S_{-y}^{\rm tot}\right] = S_y^{\rm tot}$. If we did these steps for both additional charges, we would have identified $\frac{c}{r}(c-r) = 6$ dynamical symmetries. However, we find a different story when we do check whether the additional charge removes any of the previous dynamical symmetries (see step 5). Introducing the charges means that the Hamiltonian now commutes with $S_{\pm x}^{\rm tot}$ and $S_{\pm y}^{\rm tot}$, and thus also commutes with $S_{\pm z}^{\rm tot}$. The three conservation laws together eliminate all six dynamical symmetries we listed above. This example contrasts with the Hubbard model where the dynamical symmetries of $S_{\pm z}^{\rm tot}$ could coexist with that of another charge that commutes with $S_{\pm z}^{\rm tot}$.

IV C. Illustration using $\mathfrak{su}(3)$

To demonstrate the second way charges can commute in our construction—from being part of the same Cartan subalgebra—we turn to $\mathfrak{su}(3)$. $\mathfrak{su}(3)$ has dimension $c = 8$ and rank $r = 2$. Thus, our system has eight charges that we can partition $\frac{c}{r} = 4$ sets of mutually commuting charges. These sets generate Cartan subalgebras. The eight charges of $\mathfrak{su}(3)$ can be represented by the Gell–Mann matrices [58], τ_i for $i = 1$ to 8.

We begin with one Cartan sublagebra of $\mathfrak{su}(3)$. For example, take the Cartan subalgebra with τ_3 and τ_8. In the three-dimensional representation of $\mathfrak{su}(3)$, these operators can be represented with

$$\tau_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \tau_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \quad (19)$$

As before, our charges will be sums over these operators on each site, $Q_1 = \sum_{j=1}^{N} \tau_3^{(j)}$ and $Q_2 = \sum_{j=1}^{N} \tau_9^{(j)}$. Using this Cartan subalgebra, we construct a Cartan-Weyl basis. This requires identifying $\frac{c-r}{2} = 3$ pairs of root vectors. We define the following operators, $A_{+1} \coloneqq \tau_1 + i\tau_2$, $A_{+2} \coloneqq \tau_4 + i\tau_5$, and $A_{+3} \coloneqq \tau_6 + i\tau_7$. These operators and their Hermitian conjugates are the root vectors. As we did in the previous example, we can construct the dynamical symmetries by taking sums over the operators on each site in our chain, $A_{\pm\beta}^{\rm tot} \coloneqq \sum_{j=1}^{N} A_{\pm\beta}^{(j)}$. Thus, we have again found the dynamical symmetries from the charges.

We could reverse this procedure to find the charges generated by these dynamical symmetries. We do this explicitly to point out an effect not observed in the $\mathfrak{su}(2)$ cases. We identify the operators,

$$Q_1 = c_1[A_{+1}, A_{-1}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (20)$$

$$Q_{\bar{2}} = c_2[A_{+2}, A_{-2}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ and} \quad (21)$$

$$Q_{\bar{3}} = c_3[A_{+3}, A_{-3}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (22)$$

Note that they are not linearly independent, since not all of the Killing forms between all pairs are zero: $(Q_1, Q_{\bar{2}}) = 3, (Q_1, Q_{\bar{3}}) = -3$, and $(Q_{\bar{2}}, Q_{\bar{3}}) = 3$. A Cartan subalgebra basis will require two charges. Thus, we sum over two of these three operators. We are free to do so in different ways. The choice that recovers the original two operators we began the procedure with is summing over charges $Q_{\bar{2}}$ and $Q_{\bar{3}}$: $Q_2 = \frac{1}{\sqrt{3}}(Q_{\bar{2}} + Q_{\bar{3}})$.

The system described above has two charges and three pairs of dynamical symmetries. We now want to introduce others charges that does not commute with the existing ones, i.e., more of the charges that generate $\mathfrak{su}(3)$. However, recall the dynamical symmetries for τ_3 and τ_8 are linear combinations of the other six Gell–Mann matrices. If the Hamiltonian now commutes with any elements from other Cartan subalgebras of $\mathfrak{su}(3)$, it will stop some combination of A_1, A_2, and A_3 from being dynamical symmetries. If the Hamiltonian commutes with all of $\mathfrak{su}(3)$'s charges, it will nullify all of the system's dynamical symmetries.

IV D. Hamiltonians

Although the primary aim of this study is not to construct Hamiltonians, we include some examples to link our findings with physically viable systems and establish a connection with research on quantum scars [48]. We start with the $\mathfrak{su}(2)$ case, illustrating how various Hamiltonians can transition from a single charge forming a Cartan subalgebra of $\mathfrak{su}(2)$ (U(1) symmetry) to three charges constituting the full algebra (SU(2) symmetry). An example of this is the Heisenberg model under an external field,

$$H_{\rm H} = \frac{B}{2}\Big(\sum_{j=1}^{N}\sigma_z^{(j)}\Big) + \frac{J}{2}\Big(\sum_{\langle j,k\rangle}\sum_{\langle\langle j,k\rangle\rangle}\sigma_x^{(j)}\sigma_x^{(k)} + \sigma_y^{(j)}\sigma_y^{(k)} + \sigma_z^{(j)}\sigma_z^{(k)}\Big), \tag{23}$$

where $\langle j,k\rangle$ indicates the sum over nearest neighbours, $\langle\langle j,k\rangle\rangle$ indicates the sum is over next-nearest neighbours, B is the strength of an external magnetic field, and J is a coupling constant. For $B \neq 0$, the system has one charge corresponding to a Cartan Subalgebra of $\mathfrak{su}(2)$ and one pair of dynamical symmetries. By setting $B = 0$, we introduce two more noncommuting charges into the system at the cost of removing the dynamical symmetries. We included the next-nearest neighbour interaction to break integrability. Alternatively, we could construct our Hamiltonian from genuine three-body interactions that are SU(2)-symmetric, such as

$$\sigma_x^{(j)}\sigma_y^{(j+1)}\sigma_z^{(j+2)} + \sigma_y^{(j)}\sigma_z^{(j+1)}\sigma_x^{(j+2)} + \sigma_z^{(j)}\sigma_x^{(j+1)}\sigma_y^{(j+2)} - \sigma_z^{(j)}\sigma_y^{(j+1)}\sigma_x^{(j+2)} - \sigma_x^{(j)}\sigma_z^{(j+1)}\sigma_y^{(j+2)} - \sigma_y^{(j)}\sigma_x^{(j+1)}\sigma_z^{(j+2)} \tag{24}$$

and again break the symmetry with an external field.

We can similarly study Hamiltonians for the $\mathfrak{su}(3)$ example. These Hamiltonians are less familiar but can be found using the procedure in Ref. [28],

$$H = \frac{J}{2}\Big(\sum_{\alpha}\sum_{\langle j,k\rangle}\sum_{\langle\langle j,k\rangle\rangle}\tau_\alpha^{(j)}\tau_\alpha^{(k)}\Big) + \frac{B_1}{2}\Big(\sum_j \tau_3^{(j)}\Big) + \frac{B_2}{2}\Big(\sum_j \tau_8^{(j)}\Big). \tag{25}$$

The noncommuting charges give way for dynamical symmetries by setting B_1 and B_2 to zero.These Hamiltonians align with the framework presented in Ref. [48], where the introduction of the external field breaks the non-Abelian symmetry and facilitates the emergence of quantum scars. Notably, the transitions in the Hamiltonian that give rise to quantum scars also result in non-thermalizing local observables. This correlation between quantum scars and non-thermalizing observables presents an avenue for future research.

V. DISCUSSION & OUTLOOK

This work addresses the unresolved question of whether noncommuting charges facilitate or obstruct thermalization, a topic that has attracted considerable attention in quantum thermodynamics and many-body physics [4, 10, 19–21, 59–66]. We show that noncommuting charges, across various systems, tend to decrease the number of local observables that comply with the Eigenstate Thermalization Hypothesis (ETH), effectively transitioning these systems from non-equilibrium to thermal equilibrium. This finding reinforces the notion that noncommuting charges are conducive to thermalization.

Our findings pave the way for future investigations, such as experimental tests, targeting two objectives. The first objective is to examine the emergence of nonstationary dynamics during the transition from noncommuting to commuting charge systems. This can initially be explored through simple implementations, such as those described in Section IV D. However, dynamical symmetries are sufficient, but not necessary conditions for non-stationary dynamics. Therefore, it is worthwhile to explore systems where charges transition between commuting and noncommuting states without dynamical symmetries. For instance, in the Heisenberg Hamiltonian context, instead of introducing an external field, one could use a coupling term like $(\sigma_x^{(j)}\sigma_y^{(j+1)} - \sigma_y^{(j)}\sigma_x^{(j+1)})$ to maintain U(1) symmetry while breaking SU(2) symmetry. Investigating such symmetry-breaking terms could elucidate the specific aspects of non-stationary dynamics driven by noncommuting charges and their relationship with dynamical symmetries.

Recently, noncommuting charges were argued to destabilize many-body localization [67], a type of nonstationary behaviour. Our results and this work thus naturally reinforce one another. A second opportunity for future work is to test whether our results explain the destabilization found in Ref. [67]. This would entail identifying whether dynamical symmetries also cause many-body localization and studying the algebraic relationship between those symmetries and the noncommuting charges.

A third opportunity for future work was presented in Sec. IV D. The Hamiltonians presented in this section to transition from commuting to noncommuting charges overlap with those presented in Ref. [48],to transition from a system with quantum scars to one with out them. Identifying a correspondence between those quantum scars and the non-thermalizing observables is another opportunity. Furthermore, the connection to scars could help explain why noncommuting charges increase average entanglement [19], with a difference that shrinks with the system size. Noncommuting charges seem to conflict with the existence of quantum scars, which typically have subthermal entanglement entropy. Noncommuting charges may be increasing average entangled by "eliminating" these less entangled states. Note that scars constitute a vanishing fraction of a systems with eigen states, which is consistent with the increase in entanglement in Ref. [19] quickly decreasing with system size.

A third avenue for future research, discussed in Section IV D, involves examining Hamiltonians that facilitate the transition from commuting to noncommuting charges. These Hamiltonians overlap with those in Ref.[48], which describe transitions from systems with quantum scars to those without. Exploring the relationship between quantum scars and non-thermalizing observables presents another opportunity. This exploration could further tie into the observed increase in average entanglement due to noncommuting charges [19]. Ref.[48] suggests that noncommuting charges reduce quantum scars, which have subthermal entanglement entropy and represent a small fraction of the system's eigenstates. This reduction in lesser-entangled states may contribute to the increased average entanglement observed with noncommuting charges.

The outlined future work aims to decipher the complex role of noncommuting charges in thermalization. On one hand, evidence suggests they hinder thermalization, as seen in their deviation from thermal state forms [10], reduced entropy production rates [9], more restricted dynamics compared to commuting charge systems [23], and hindering derivations of the thermal state's form [5–8]. On the other hand, evidence supports their role in promoting thermalization, indicated by increased average entanglement [19, 20] and the elimination of nonstationary dynamics. This dichotomy suggests that noncommuting charges have a more nuanced relation with thermalization, necessitating further research to fully understand their influence.

ACKNOWLEDGMENTS

SM would like to thank Berislav Buča and Abhinav Prem for useful discussions and Nicole Yunger Halpern for asking him a question that ultimately motivated this work—what do dynamical symmetries have to do with noncommuting charges? Further thanks are extended to Matthew Graydon, Raymond Laflamme, Zachary Mann, José Polo-Gómez, Abhinav Prem, and Nicole Yunger Halpern for their comments on the manuscript. This work received support from the Vanier C.G.S.

REFERENCES

[1] M. Srednicki, Chaos and quantum thermalization, Physical Review E **50**, 888 (1994).
[2] J. M. Deutsch, Quantum statistical mechanics in a closed system, Physical review a **43**, 2046 (1991).
[3] M. Rigol, V. Dunjko, and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, Nature **452**, 854 (2008).
[4] S. Majidy, W. F. Braasch Jr, A. Lasek, T. Upadhyaya, A. Kalev, and N. Yunger Halpern, Noncommuting conserved charges in quantum thermodynamics and beyond, Nature Reviews Physics **5**, 689 (2023).
[5] N. Yunger Halpern, P. Faist, J. Oppenheim, and A. Winter, Microcanonical and resource-theoretic derivations of the thermal state of a quantum system with noncommuting charges, Nature communications **7**, 1 (2016).
[6] Y. Guryanova, S. Popescu, A. J. Short, R. Silva, and P. Skrzypczyk, Thermodynamics of quantum systems with multiple conserved quantities, Nature communications **7**, 1 (2016).
[7] M. Lostaglio, D. Jennings, and T. Rudolph, Thermodynamic resource theories, non-commutativity and maximum entropy principles, New Journal of Physics **19**, 043008 (2017).
[8] N. Yunger Halpern, Beyond heat baths ii: Framework for generalized thermodynamic resource theories, Journal of Physics A: Mathematical and Theoretical **51**, 094001 (2018).
[9] G. Manzano, J. M. Parrondo, and G. T. Landi, Non-abelian quantum transport and thermosqueezing effects, PRX Quantum **3**, 010304 (2022).
[10] C. Murthy, A. Babakhani, F. Iniguez, M. Srednicki, and N. Yunger Halpern, Non-abelian eigenstate thermalization hypothesis, arXiv preprint arXiv:2206.05310 (2022).
[11] D. Sen, The uncertainty relations in quantum mechanics, Current Science , 203 (2014).
[12] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Entropic uncertainty relations and their applications, Reviews of Modern Physics **89**, 015002 (2017).
[13] C. J. Fewster and R. Verch, Quantum fields and local measurements, Communications in Mathematical physics **378**, 851 (2020).
[14] J. Polo-Gómez, L. J. Garay, and E. Martín-Martínez, A detector-based measurement theory for quantum field theory, Physical Review D **105**, 065003 (2022).
[15] A. Aspect, Bell's inequality test: more ideal than ever, Nature **398**, 189 (1999).
[16] C. Emary, N. Lambert, and F. Nori, Leggett–garg inequalities, Reports on Progress in Physics **77**, 016001 (2013).
[17] S.-S. Majidy, H. Katiyar, G. Anikeeva, J. Halliwell, and R. Laflamme, Exploration of an augmented set of leggett-garg inequalities using a noninvasive continuous-in-time velocity measurement, Physical Review A **100**, 042325 (2019).
[18] S. Majidy, J. J. Halliwell, and R. Laflamme, Detecting violations of macrorealism when the original leggett-garg inequalities are satisfied, Physical Review A **103**, 062212 (2021).
[19] S. Majidy, A. Lasek, D. A. Huse, and N. Yunger Halpern, Non-abelian symmetry can increase entanglement entropy, Physical Review B **107**, 045102 (2023).
[20] S. Majidy, U. Agrawal, S. Gopalakrishnan, A. C. Potter, R. Vasseur, and N. Yunger Halpern, Critical phase and spin sharpening in su (2)-symmetric monitored quantum circuits, Physical Review B **108**, 054307 (2023).

[21] T. Upadhyaya, W. F. Braasch Jr, G. T. Landi, and N. Yunger Halpern, What happens to entropy production when conserved quantities fail to commute with each other, arXiv preprint arXiv:2305.15480 (2023).
[22] G. Manzano, Squeezed thermal reservoir as a generalized equilibrium reservoir, Physical Review E **98**, 042123 (2018).
[23] I. Marvian, Restrictions on realizable unitary operations imposed by symmetry and locality, Nature Physics **18**, 283 (2022).
[24] J. Preskill, Reliable quantum computers, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences **454**, 385 (1998).
[25] D. Gottesman, An introduction to quantum error correction and fault-tolerant quantum computation, in *Quantum information science and its contributions to mathematics, Proceedings of Symposia in Applied Mathematics*, Vol. 68 (2010) pp. 13–58.
[26] E. T. Campbell, B. M. Terhal, and C. Vuillot, Roads towards fault-tolerant universal quantum computation, Nature **549**, 172 (2017).
[27] N. Yunger Halpern, M. E. Beverland, and A. Kalev, Noncommuting conserved charges in quantum many-body thermalization, Physical Review E **101**, 042117 (2020).
[28] N. Yunger Halpern and S. Majidy, How to build hamiltonians that transport noncommuting charges in quantum thermodynamics, npj Quantum Information **8**, 10 (2022).
[29] D. Loss and D. P. DiVincenzo, Quantum computation with quantum dots, Physical Review A **57**, 120 (1998).
[30] H.-S. Zhong, H. Wang, Y.-H. Deng, M.-C. Chen, L.-C. Peng, Y.-H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu, *et al.*, Quantum computational advantage using photons, Science **370**, 1460 (2020).
[31] T. Rudolph and S. S. Virmani, The two-qubit singlet/triplet measurement is universal for quantum computing given only maximally-mixed initial states, Nature Communications **14**, 7800 (2023).
[32] M. H. Freedman, M. B. Hastings, and M. S. Zini, Symmetry protected quantum computation, Quantum **5**, 554 (2021).
[33] B. Buča, J. Tindall, and D. Jaksch, Non-stationary coherent quantum many-body dynamics through dissipation, Nature Communications **10**, 1730 (2019).
[34] M. Medenjak, B. Buča, and D. Jaksch, Isolated heisenberg magnet as a quantum time crystal, Physical Review B **102**, 041117 (2020).
[35] M. Medenjak, T. Prosen, and L. Zadnik, Rigorous bounds on dynamical response functions and time-translation symmetry breaking, SciPost Physics **9**, 003 (2020).
[36] K. Chinzei and T. N. Ikeda, Time crystals protected by floquet dynamical symmetry in hubbard models, Physical Review Letters **125**, 060601 (2020).
[37] S. Moudgalya, N. Regnault, and B. A. Bernevig, η-pairing in hubbard models: From spectrum generating algebras to quantum many-body scars, Physical Review B **102**, 085140 (2020).
[38] A. Bohm, Y. Ne'eman, and A. O. Barut, *Dynamical Groups And Spectrum Generating Algebras*, Vol. 2 (World Scientific, 1988).
[39] B. Gruber and T. Otsuka, *Symmetries in science VII: spectrum-generating algebras and dynamic symmetries in physics* (Springer Science & Business Media, 2012).
[40] J. Tindall, C. S. Muñoz, B. Buča, and D. Jaksch, Quantum synchronisation enabled by dynamical symmetries and dissipation, New Journal of Physics **22**, 013026 (2020).
[41] B. Buča and D. Jaksch, Dissipation induced nonstationarity in a quantum gas, Physical review letters **123**, 260401 (2019).
[42] B. Buča, Out-of-time-ordered crystals and fragmentation, Physical Review Letters **128**, 100601 (2022).
[43] B. Buca, A. Purkayastha, G. Guarnieri, M. T. Mitchison, D. Jaksch, and J. Goold, Quantum many-body attractors, arXiv preprint arXiv:2008.11166 (2020).
[44] E. J. Heller, Bound-state eigenfunctions of classically chaotic hamiltonian systems: scars of periodic orbits, Physical Review Letters **53**, 1515 (1984).
[45] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, Weak ergodicity breaking from quantum many-body scars, Nature Physics **14**, 745 (2018).
[46] M. Serbyn, D. A. Abanin, and Z. Papić, Quantum many-body scars and weak breaking of ergodicity, Nature Physics **17**, 675 (2021).
[47] S. Moudgalya, B. A. Bernevig, and N. Regnault, Quantum many-body scars and hilbert space fragmentation: a review of exact results, Reports on Progress in Physics **85**, 086501 (2022).
[48] N. O'Dea, F. Burnell, A. Chandran, and V. Khemani, From tunnels to towers: Quantum scars from lie algebras and q-deformed lie algebras, Physical Review Research **2**, 043305 (2020).
[49] Z. Zhang, J. Tindall, J. Mur-Petit, D. Jaksch, and B. Buča, Stationary state degeneracy of open quantum systems with non-abelian symmetries, Journal of Physics A: Mathematical and Theoretical **53**, 215304 (2020).
[50] A. Das and S. Okubo, *Lie groups and Lie algebras for physicists* (World Scientific, 2014).
[51] F. Iachello, *Lie algebras and applications*, Vol. 12 (Springer, 2006).
[52] R. Gilmore, *Lie groups, Lie algebras, and some of their applications* (Courier Corporation, 2012).
[53] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Vol. 9 (Springer Science & Business Media, 2012).
[54] B. C. Hall, *Lie groups, Lie algebras, and representations* (Springer, 2013).
[55] R. Campoamor-Stursberg and M. R. De Traubenberg, *Group theory in physics: a practitioner's guide* (World Scientific, 2019).
[56] C. N. Yang, η pairing and off-diagonal long-range order in a hubbard model, Physical review letters **63**, 2144 (1989).
[57] D. Jakubczyk and P. Jakubczyk, On the su (2)× su (2) symmetry in the hubbard model, Open Physics **10**, 906 (2012).
[58] R. N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Dover, 2006).
[59] F. Kranzl, A. Lasek, M. K. Joshi, A. Kalev, R. Blatt, C. F. Roos, and N. Yunger Halpern, Experimental observation of

thermalisation with noncommuting charges, arXiv preprint arXiv:2202.04652 (2022).
[60] J. D. Noh, Eigenstate thermalization hypothesis in two-dimensional x x z model with or without su (2) symmetry, Physical Review E **107**, 014130 (2023).
[61] J. Siegl and J. Schliemann, Imperfect many-body localization in exchange-disordered isotropic spin chains, New Journal of Physics **25**, 123002 (2023).
[62] F. Ares, S. Murciano, E. Vernier, and P. Calabrese, Lack of symmetry restoration after a quantum quench: an entanglement asymmetry study, SciPost Physics **15**, 089 (2023).
[63] J. Tabanera-Bravo, J. M. Parrondo, M. Esposito, and F. Barra, Thermalization and dephasing in collisional reservoirs, Physical Review Letters **130**, 200402 (2023).
[64] I. Marvian, Theory of quantum circuits with abelian symmetries, arXiv preprint arXiv:2302.12466 (2023).
[65] B. Dabholkar and F. Alet, Ergodic and non-ergodic properties of disordered su (3) chains, arXiv preprint arXiv:2403.00442 (2024).
[66] L. P. García-Pintos, K. Bharti, J. Bringewatt, H. Dehghani, A. Ehrenberg, N. Yunger Halpern, and A. V. Gorshkov, Estimation of hamiltonian parameters from thermal states, arXiv preprint arXiv:2401.10343 (2024).
[67] A. C. Potter and R. Vasseur, Symmetry constraints on many-body localization, Physical Review B **94**, 224206 (2016).