

Superconducting antiqubits achieve optimal phase estimation via unitary inversion

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A positron is equivalent to an electron traveling backward through time. Casting transmon superconducting qubits as akin to electrons, we simulate a positron with a transmon subject to particular resonant and off-resonant drives. We call positron-like transmons “antiqubits.” An antiqubit’s effective gyromagnetic ratio equals the negative of a qubit’s. This fact enables us to time-invert a unitary implemented on a transmon by its environment. We apply this platform-specific unitary inversion, with qubit–antiqubit entanglement, to achieve a quantum advantage in phase estimation: consider measuring the strength of a field that points in an unknown direction. An entangled qubit–antiqubit sensor offers the greatest possible sensitivity (amount of Fisher information), per qubit, per application of the field. We prove this result theoretically and observe it experimentally. This work shows how antimatter, whether real or simulated, can enable platform-specific unitary inversion and benefit quantum information processing.

One can regard a positron as an electron moving backward in time. This insight has impacted fundamental physics, as championed by Wheeler, Feynman, and Stückelberg [1–4]. Can one apply the insight for practical benefit? One practical challenge, common across quantum computation and metrology, is *unitary inversion*. Consider a quantum system subject to an external field that effects the unitary operation $U_\alpha := e^{-i\alpha H}$. (We set $\hbar = 1$.) Often, one wishes to implement $U_\alpha^\dagger$. Such unitary inversion can help one detect quantum-information scrambling [5–7], superpose time evolutions [8], implement higher-order quantum transformations (e.g., evolutions of quantum channels) [9, 10], measure out-of-time-order correlators [11, 12], and support quantum singular-value transformations [13, 14].

Conventional unitary inversion can operate imprecisely and requires substantial resources. In a naïve approach, one infers the form of U_α from process tomography [15–18]. Then, one constructs a setup intended to apply $U_\alpha^\dagger$. Alternatively, one can implement $U_\alpha^\dagger$ with a unitary-reversal algorithm [9, 19]. Such an algorithm costs many U_α applications, which serve as resources in quantum computing and metrology: let d denote the system-of-interest Hilbert space’s dimensionality. The algorithms require $\mathcal{O}(d^2)$ applications of U_α per $U_\alpha^\dagger$ implementation, like tomography.

We propose to invert unitaries by applying the positron insight above. The electron’s gyromagnetic ratio, γ , helps determine the electron’s magnetic moment and hence the electron’s coupling to an external field and hence α . The positron has a gyromagnetic ratio $-\gamma$. If exposed to the same field as an electron, a positron undergoes $e^{-i(-\alpha)H} = U_\alpha^\dagger$. Hence one can effectively invert a unitary, and so effectively reverse time, by exchanging matter with antimatter.

We experimentally simulate an exchange of a transmon with antimatter. The strategy relies on resonant and off-resonant drives, which negate the effective gyromagnetic ratio of the transmon’s pseudospin. This negation inverts the unitary to which a magnetic field subjects the transmon, regardless of the field’s direction. We call this process *platform-specific unitary inversion*. Also, we call positron-like transmons *antiqubits*.

We apply platform-specific unitary inversion and entanglement to achieve a quantum advantage in sensing. Consider a field pointing in an arbitrary direction, which could be unknown. One measures the field strength α in *quantum phase estimation*, a task prevalent in quantum algorithms and metrology [20–29]. To accomplish this task, we entangle a qubit transmon with an antiqubit. We call the entangled pair *synthetic positronium*, after the positronium atom formed from a bound-together electron and positron. The field applies U_α to the qubit and, via platform-specific unitary inversion, $U_\alpha^\dagger$ to the antiqubit. This inversion boosts the Fisher information (FI) obtainable, or sensitivity achievable, about the field. We quantify the resources required using the *space–time volume* v_{st} , defined as (number of transmons used) $\times$ (number of sequential applications of the unitary) [30, 31]. Our strategy achieves the greatest possible amount of FI per two units of space–time volume, 4. [We fix the space–time volume to be $v_{\text{st}} = 2$ throughout this proof-of-principle work for simplicity. However, an extension of our strategy achieves the maximum possible Fisher information, regardless of the space–time volume (Sec. VI)]. Experimentally, we obtain an FI of 3.02 per two units of space–time volume. In contrast, two entangled qubits can achieve 1; and a separable qubit–antiqubit state, 4/3, on average over trials. Hence entanglement and platform-specific unitary inversion, realized

with antimatter simulation, enable a quantum advantage in metrology. Furthermore, we prove, that entanglement and effective unitary inversion enable the unique strategy that achieves the optimal FI per two units of space–time volume.

The rest of this paper is organized as follows. Section I provides background about phase estimation. Section II theoretically introduces positronium metrology facilitated by platform-specific unitary inversion. In Sec. III, we describe the experimental setup. Section IV explains how we perform platform-specific unitary inversion. The experimental results follow in Sec. V. Section VI outlines opportunities established by this work.

I. BACKGROUND: PHASE ESTIMATION

This section introduces a basic task in quantum metrology, phase estimation. Below, we specify the task, then review common metrics for evaluating success: the FI and quantum Fisher information (QFI). We illustrate with a qubit example and three approaches to it. We compare the approaches using a metric that reflects not only the information gained metrologically, but also the resources spent: the space–time volume.

The following task exemplifies phase estimation. Let U_α denote an arbitrary unitary parameterized by $\alpha \in \mathbb{R}$. To estimate α , we proceed as in Fig. 1(a): we prepare a probe in $|\psi\rangle$. The unitary maps $|\psi\rangle$ to $U_\alpha |\psi\rangle =: |\psi_\alpha\rangle$. We measure some basis $\{|j\rangle\}$ of the probe. Outcome j occurs with a probability $P_j = |\langle j|\psi_\alpha\rangle|^2$.

The FI I_α quantifies the probability distribution’s sensitivity to changes in α : $I_\alpha := \sum_j (\partial_\alpha P_j)^2 / P_j$. Maximizing the FI over measurements yields the QFI, $\mathcal{I}_\alpha$ [32, 33]:

$$I_\alpha \leq \mathcal{I}_\alpha = 4 \left(\langle \partial_\alpha \psi_\alpha | \partial_\alpha \psi_\alpha \rangle - |\langle \psi_\alpha | \partial_\alpha \psi_\alpha \rangle|^2 \right). \quad (1)$$

The QFI, too, obeys an upper bound: suppose that $U_\alpha = e^{-i\alpha H}$. Denote by ΔH the difference between the generator’s greatest eigenvalue, $E_{\max}$, and the least, $E_{\min}$: $\Delta H := E_{\max} - E_{\min}$. The QFI obeys

$$\mathcal{I}_\alpha \leq \max_{|\psi\rangle} \left\{ \langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2 \right\} = (\Delta H)^2. \quad (2)$$

The QFI saturates this bound if $|\psi\rangle$ is an equal-weight superposition of an eigenvalue- $E_{\max}$ eigenstate and an eigenvalue- $E_{\min}$ eigenstate.

We illustrate with a qubit subject to an external field. Denote by $\hat{\mathbf{n}} = n_x \hat{\mathbf{x}} + n_y \hat{\mathbf{y}} + n_z \hat{\mathbf{z}}$ the field’s direction; and, by $\boldsymbol{\sigma} = X \hat{\mathbf{x}} + Y \hat{\mathbf{y}} + Z \hat{\mathbf{z}} \equiv \sigma_x \hat{\mathbf{x}} + \sigma_y \hat{\mathbf{y}} + \sigma_z \hat{\mathbf{z}}$, a vector of the Pauli operators. For future reference, we denote by $|a\pm\rangle$ the σ_a eigenstate associated with the eigenvalue ± 1 . Also, we define $|0\rangle := |z+\rangle$ and $|1\rangle := |z-\rangle$. The unitary $U_\alpha = e^{-i\alpha \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} / 2}$ effects a rotation whose angle α we wish to estimate. Since the generator $H = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} / 2$, the QFI is $(\Delta H)^2 = 1$.

We now present three strategies for inferring about α . First, suppose we know the rotation axis $\hat{\mathbf{n}}$. We can

achieve the QFI by preparing even-weight superpositions of the H eigenstates.

Second, suppose we do not know $\hat{\mathbf{n}}$ and can use only one qubit per trial. We lack information about three parameters: α and the two angles that specify $\hat{\mathbf{n}}$. In contrast, only two parameters specify a pure qubit state [34]. Due to this discrepancy, we cannot necessarily estimate α from copies of just one qubit state [35]: if we are unlucky, the Bloch vector points along $\pm \hat{\mathbf{n}}$. Consequently, the field cannot rotate the qubit, which acquires no information about the field. The optimal strategy is to prepare probes in X eigenstates in one batch of trials, Y eigenstates in another batch, and Z eigenstates in a third batch. On average over trials, this strategy achieves the QFI, $2/3 < 1$ [35].

Third, suppose that we can use two qubits per trial. *Agnostic phase estimation* leverages entanglement to boost the QFI, which is again achievable [35]. Figure 1(b) illustrates the protocol: we prepare a probe qubit and an ancilla qubit in the singlet $|\Psi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Then, U_α acts on the probe. We measure the positive-operator-valued measure (POVM) [36] $\{|\Psi^-\rangle\langle\Psi^-|, \mathbb{1} - |\Psi^-\rangle\langle\Psi^-|\}$. The greater the α , the more U_α perturbs the joint state away from $|\Psi^-\rangle$, and the greater our probability of observing the $|\Psi^-\rangle\langle\Psi^-|$ outcome. This strategy yields an FI equal to the QFI, 1. Agnostic sensing requires no knowledge of $\hat{\mathbf{n}}$ and requires only one U_α application per trial—but requires two qubits per trial.

To compare metrological schemes comprehensively, we need a resource measure that incorporates not only FI, but also resource costs. Different schemes may require different numbers of unitary applications, and different numbers of particles, per trial. Hence we invoke Feynman’s *space–time volume*, v_{st} [30]. It draws inspiration from classical complexity theory: the total resource tracked there is the product of memory and runtime [31]. In quantum computing, an algorithm’s space–time volume equals the number of qubits times the number of circuit layers. Here, the space–time volume equals the number of qubits used, times the number of sequential U_α applications, per trial [37]. We compare protocols’ FIs per $v_{\text{st}} = 2$ units of space–time volume, as some protocols require two qubits each.

Let us apply this metric to phase estimation in the absence of knowledge about $\hat{\mathbf{n}}$. One qubit can yield an average FI, per two units of space–time volume, of $I_\alpha = 4/3$. Agnostic sensing yields only 1. Incorporating qubits into the resource cost, we lower the agnostic-sensing efficacy below the single-qubit strategy’s.

We show next how to outperform both these metrological strategies with positronium metrology. It achieves an I_α of 4 per two units of space–time volume, saturating the theoretical limit on this metric [38]. Only effective unitary inversion, combined with a singlet, can achieve this FI rate [38]. Our approach relies on entanglement and on platform-specific unitary inversion applied to a qubit–antiqubit pair.

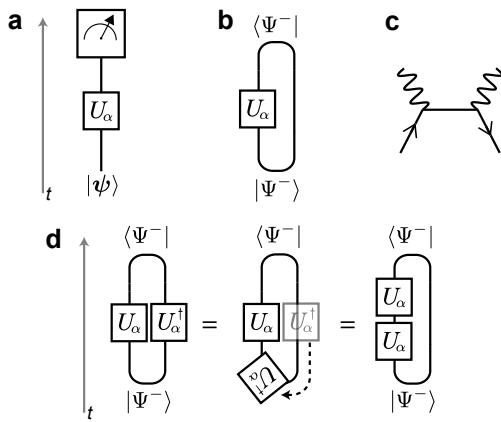


FIG. 1. **Quantum phase estimation, including platform-specific unitary inversion implemented with synthetic positronium.** (a) Circuit diagram for basic phase estimation. Time runs vertically. One prepares a probe in a state $|\psi\rangle$, evolves it under a unitary U_α , and measures the probe. (b) Agnostic phase estimation employs a two-qubit entangled state to simulate a closed timelike curve. The $\cup$ represents singlet preparation; and the $\cap$, a measurement of whether the state remains a singlet. (c) Read from bottom to top, this diagram represents electron–positron scattering that produces photons. One can view the positron as an electron moving backward through time, as suggested by the arrows’ orientations. We leverage this insight to perform platform-specific unitary inversion and positronium metrology. (d) Positronium metrology involves a qubit probe that undergoes U_α and an antiquubit ancilla that, due to platform-specific unitary inversion, undergoes $U_\alpha^\dagger$ (leftmost diagram). Imagine sliding the $U_\alpha^\dagger$ box leftward along the wire (central diagram). When passing through the $\cup$, the $U_\alpha^\dagger$ is inverted. Hence positronium metrology is equivalent to implementing U_α^2 on the probe (rightmost diagram). Doubling the number of U_α s applied, relative to agnostic sensing, doubles the amount of FI achievable per two units of space–time volume.

II. THEORY OF POSITRONIUM METROLOGY

Using positronium metrology, one can achieve the optimal FI about α per two units of space–time volume, without knowing $\hat{n}$. The scheme leverages two distinct processes that are mathematically equivalent to time reversals (two *effective* time reversals). We introduce the two sequentially, then assess the scheme’s efficacy.

The first effective time reversal results from entanglement manipulation and enables agnostic phase estimation. The previous section reviewed agnostic phase estimation [35], depicted in Fig. 1(b). We can read that figure in two ways. Traversed from bottom to top, the figure shows (i) a probe qubit and ancilla qubit prepared in a singlet, (ii) the probe undergoing a unitary, and (iii) a measurement of whether the qubits remain in a singlet. Alternatively, we can interpret the figure as one qubit’s worldline. The worldline forms a loop, or closed timelike curve [39–42]: the qubit reverses temporal direction, relative to the laboratory rest frame, at the $\cup$ and $\cap$.

(The qubit also undergoes a transformation, discussed below, at each turning point.) These symbols represent entanglement manipulations in the first reading of the figure. The entanglement manipulations—and the effective time reversals—enable agnostic phase estimation as follows. When performing agnostic phase estimation, we do not know the field direction $\hat{n}$. Hence we do not know the optimal state in which to prepare the probe at the beginning of the protocol. We effectively teleport that state backward in time (from the diagram’s top to its bottom), via the entanglement manipulations—by effectively reversing time.

During agnostic phase estimation, the field acts only on the probe. Can we boost the FI by acting the field on the ancilla? Doing so naïvely would transform the singlet identically: $|\Psi^-\rangle \mapsto (U_\alpha \otimes U_\alpha)|\Psi^-\rangle = |\Psi^-\rangle$. The second U_α undoes the first one’s effect, due to the singlet’s rotational invariance under tensor products of identical unitaries.

Instead, we propose, we should subject the ancilla to $U_\alpha^\dagger$. We do so via the second effective time reversal: we convert the ancilla into an antiqubit by negating its effective gyromagnetic ratio. This negation negates α , subjecting the (antiqubit) ancilla to $e^{-i(-\alpha)\hat{n}\cdot\sigma/2} = U_\alpha^\dagger$. This strategy, we call *positronium metrology*. Figure 1(c) illustrates the fundamental insight that we leverage: interchanging matter with antimatter effectively reverses time.

Figure 1(d) illustrates how positronium metrology boosts the FI. The leftmost diagram shows the qubit probe undergoing U_α and the antiqubit ancilla undergoing $U_\alpha^\dagger$. Consider sliding the $U_\alpha^\dagger$ box downward along the wire, as in the central diagram. When the box passes through the $\cup$, it undergoes a universal NOT (for the central diagram to represent the same physics as the leftmost) [35]: $U_\alpha^\dagger = e^{i\alpha\hat{n}\cdot\sigma/2} \mapsto e^{-i\alpha\hat{n}\cdot\sigma/2} = U_\alpha$. Implementing positronium metrology, therefore, is equivalent to implementing U_α^2 on the probe [rightmost diagram in Fig. 1(d)]. However, truly implementing U_α^2 costs two units of space–time volume, whereas our scheme costs one unit. The effective U_α^2 doubles the FI gleaned about α : positronium metrology achieves $I_\alpha = 4$ at the cost of $v_{\text{st}} = 2$. This FI equals the corresponding QFI, which is the greatest possible QFI achievable with any strategy that consumes ≤ 2 units of space–time volume, even if one knows $\hat{n}$ [38].

III. EXPERIMENTAL SETUP

Our experiment features three transmons: a qubit q that acts as ordinary matter, an antiqubit $\bar{q}$, and a coupler c . (We avoid the phrases *transmon qubit* and *superconducting qubit*, reserving the word *qubit* for q .) The following frequencies specify the gaps between transmons’ ground and first-excited energy levels at zero flux (in the absence of drives): $\omega_q/(2\pi) = 4.167$ GHz, $\omega_{\bar{q}}/(2\pi) = 4.274$ GHz, and $\omega_c/(2\pi) = 5.250$ GHz. We modulate ω_c

to activate parametric interactions between q and $\bar{q}$.

We prepare the singlet via the following steps. Each transmon begins in $|z+\rangle$. We subject q to a π -pulse, preparing it in $|z-\rangle$. Next, we effect a $\sqrt{i\text{SWAP}}$ gate, by implementing the resonant parametric interaction for 104 ns. Finally, we correct the antiqubit's phase [38]. This process prepares the singlet with a fidelity of 97 %.

Similarly, we measure $\{|\Psi^-\rangle\langle\Psi^-|, \mathbb{1} - |\Psi^-\rangle\langle\Psi^-|\}$ in two steps. First, we perform a $\sqrt{i\text{SWAP}}$. Then, we measure each qubit's computational basis. We achieve single-qubit measurement fidelities of ≈ 95 %, for which we correct throughout this work.

In the absence of extraneous fields (fields other than the quantization field), we set each transmon's Hamiltonian $\propto Z$, in accordance with convention. Resonant drives effect controlled rotations about the x - and y -axes. Therefore, transmon $j \in \{q, \bar{q}\}$ has a Hamiltonian expressible in terms of effective-magnetic-field components Ω_x , Ω_y and δ_j :

$$H_j = \Omega_x X + \Omega_y Y + \delta_j Z. \quad (3)$$

IV. PLATFORM-SPECIFIC UNITARY INVERSION

Figure 2(a) sketches transmon-specific unitary inversion. The qubit and antiqubit pass through the same magnetic field (green arrows). The qubit undergoes U_α . The antiqubit undergoes $U_\alpha^\dagger$, having an equal-magnitude, opposite-sign effective gyromagnetic ratio. We effect this gyromagnetic ratio via two control techniques, depicted in Fig. 2(b): (i) We apply a Z gate before subjecting $\bar{q}$ to the Hamiltonian (3), then apply a Z gate afterward. The gates effectively invert the magnetic field's x - and y -components. (ii) We apply a single-frequency drive to both transmons. It induces an alternating-current (AC) Stark shift of δ_q to the qubit and a shift $\delta_{\bar{q}} = -\delta_q$ to the antiqubit. This strategy effectively inverts the z -component of the magnetic field experienced by $\bar{q}$. We now elaborate on these techniques sequentially.

Z gates effectively invert the magnetic field's x - and y -components. To show how, we leverage the Taylor expansion of $U_\alpha = e^{-i\alpha\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2}$ and the Pauli operators' commutation relations:

$$Ze^{-i\alpha\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2}Z = e^{-i\alpha\hat{\mathbf{n}}\cdot(Z\boldsymbol{\sigma}Z)/2} \quad (4)$$

$$= e^{-i\alpha(-n_x X - n_y Y + n_z Z)/2}. \quad (5)$$

Experimentally, we implement a Z gate by physically rotating the qubit through an angle π about the z -axis. Two on-resonance microwave pulses effect each Z gate [38].

Figure 2(c) demonstrates this strategy's efficacy. We constructed the figure from the following two protocols, the first applied to q and the second applied to $\bar{q}$: in each of many trials, we prepared q in $|+y\rangle$. We set $\Omega_x/(2\pi)$

to 13.2 MHz for a time τ . In each of many other trials, to simulate antimatter, we performed Z gates before and after the Ω_x pulse. Figure 2(c) shows the transmons' $\langle Z \rangle$ s plotted against time. The square blue markers represent qubit data; and the circular green markers, antiqubit data. $\bar{q}$ rotates oppositely q , as expected. The blue diamonds and green triangles follow from analogous experiments that begin with $|+x\rangle$ and involve rotations about the y -axis.

We have shown how to effectively invert a field's x - and y - components, using Z gates implemented with resonant drives. Using an off-resonant microwave drive, we can effectively invert a field's z -component. We drive both transmons simultaneously. They begin with different energy gaps and so experience different AC Stark shifts: the qubit's gap changes by an amount δ_q ; and the antiqubit's, by $\delta_{\bar{q}}$. Figure 2(e) shows δ_q and $\delta_{\bar{q}}$ versus drive frequency. At the magic frequency 4.177 GHz, $\delta_{\bar{q}} = -\delta_q$; the qubit and antiqubit undergo opposite z -rotations. Figure 2(f) shows δ_q and $\delta_{\bar{q}}$ versus drive amplitude. This figure confirms that one drive frequency induces equal-magnitude, opposite-sign AC Stark shifts on the transmons.

V. POSITRONIUM-METROLOGY EXPERIMENT

Figure 3 details the phase-estimation experiment enhanced by synthetic positronium and platform-specific unitary inversion. Below, we detail the protocol. Then, we analyze the possible measurement outcomes' probabilities. From the probabilities, we calculate the FI per two units of space-time volume. We then compare positronium metrology with a protocol that features q and $\bar{q}$ but no entanglement.

The circuit diagram 3(a) shows our positronium-metrology protocol's three steps: first, we prepare q and $\bar{q}$ in a singlet. Second, we implement U_α on q and $U_\alpha^\dagger$ on $\bar{q}$, using platform-specific unitary inversion. Third, we measure the POVM $\{|\Psi^-\rangle\langle\Psi^-|, \mathbb{1} - |\Psi^-\rangle\langle\Psi^-|\}$. Let $P(|\Psi^-\rangle)$ denote the probability of obtaining the outcome associated with the first projector.

Figure 3(b) shows $P(|\Psi^-\rangle)$ versus α . Different colors and marker shapes correspond to different rotation axes $\hat{\mathbf{n}} = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. Regardless of $\hat{\mathbf{n}}$, $P(|\Psi^-\rangle) \approx \cos(2\alpha)$. The 2 comes from the equivalence between (i) positronium metrology and (ii) evolving the qubit under U_α^2 (Sec. II). Two practicalities limit the $P(|\Psi^-\rangle)$ curves' contrasts and so the experimentally inferable FI. One practicality consists of the state-preparation and measurement fidelities. The other practicality affects just the $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ data: if $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, we apply an AC Stark tone, which is not far off resonance. It therefore rotates the transmons slightly about the x - and y -axes [38].

We can calculate the experimentally achievable FI from Fig. 3(b). First, we fit a curve to each experimental dataset. Then, we identify the α value at which the slope maximizes. Figure 3(e) shows the resulting FI. Different

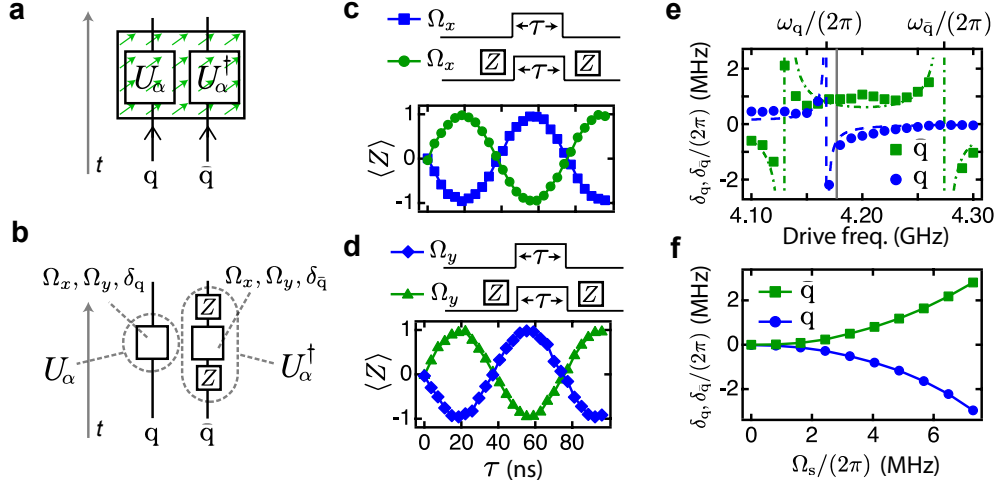


FIG. 2. **Platform-specific unitary inversion.** (a) Qubit and antiqubit transmons. An external field (green arrows) effects the unitary U_α on q and $U_\alpha^\dagger$ on $\bar{q}$. (b) We simulate antimatter by applying two techniques. q undergoes U_α due to a pseudomagnetic field with components Ω_x , Ω_y , and δ_q . The antiqubit undergoes $U_\alpha^\dagger$. The platform-specific unitary inversion results from Z gates and from the field components Ω_x , Ω_y , and $\delta_{\bar{q}}$. (c) The Z gates effectively invert the external-field component Ω_x . To check, we prepare $|y+\rangle$, then apply the field for a time τ . In half the trials, we apply a Z gate before the field application and another Z afterward. (d) Same as (c), except with Ω_y and $|x+\rangle$. (e) An off-resonant drive generates the external field's z -component. The drive induces an AC Stark shift of δ_q on the qubit and $\delta_{\bar{q}}$ on the antiqubit. Dashed lines represent theoretical predictions. Points represent frequency shifts measured with Ramsey measurements. At 4.177 GHz (solid gray line), $\delta_q = -\delta_{\bar{q}}$. The qubit has a transition frequency $\omega_q/(2\pi)$; and the antiqubit, one of $\omega_{\bar{q}}/(2\pi)$. (f) AC Stark shifts versus drive amplitude Ω_s [38].

abscissas correspond to rotations about the x -, y -, and z -axes. On average over the axes, the FI is 3.03 ± 0.07 per two units of phase-space volume. The uncertainty, $\delta = 0.07$, arises from curve fitting. It equals the geometric mean, $\delta = \frac{1}{3} \sqrt{\delta_x^2 + \delta_y^2 + \delta_z^2}$, of the standard deviations $\delta_{x,y,z}$ in the curve parameters associated with different rotational axes.

To highlight positronium metrology's entanglement advantage, we compare our strategy with phase estimation that leverages an unentangled qubit-antiquubit sensor. The competitor strategy provides the greatest possible FI achievable if, as in positronium sensing, all trials must begin with identical state preparations [43]. Figure 3(c) shows the circuit diagram: we prepare q in $|x+\rangle$ and $\bar{q}$ in $|z+\rangle$. Then, U_α evolves q , while $U_\alpha^\dagger$ evolves $\bar{q}$. Finally, we measure the qubit's $\{|x+\rangle\langle x+|, |x-\rangle\langle x-|\}$, obtaining the first outcome with a probability $P(|x+\rangle)$. We simultaneously measure the antiqubit's $\{|z+\rangle\langle z+|, |z-\rangle\langle z-|\}$, associated with an analogous $P(|z+\rangle)$.

We analyze the competitor protocol as follows. For technical reasons, we assume that the metrologist learns $\hat{n}$ after the experiment [44]. This assumption gives the competitor an advantage over positronium sensing. Nevertheless, we will see, positronium sensing achieves a greater I_α/v_{st} . Figure 3(d) shows the $P(|x+\rangle)$ and $P(|z+\rangle)$ versus α for each of three rotation axes. If q rotates about the y - or z -axis, the qubit's $P(|x+\rangle) \approx \cos(\alpha)$. If q rotates about the x -axis, the qubit's

$P(|x+\rangle) = 1$. This probability's α -independence implies that we can learn nothing about the field. Analogous statements concern the antiqubit. Figure 3(e) reports the inferable FI per two units of space-time volume. The strategy achieves an FI of $I_\alpha = 1.3 \pm 0.1$, at a cost of $v_{\text{st}} = 2$, on average over the three axes, slightly below the theoretically predicted $4/3$ [35] and below the positronium-sensing value, 3.03 ± 0.07 . The positronium-sensing and entanglement-free strategies outperform agnostic sensing performed without antimatter [35]. In the latter experiment, a qubit sensor, entangled with a qubit ancilla, achieved an FI of $I_\alpha = 0.72 \pm 0.03$, while expending $v_{\text{st}} = 2$.

VI. OUTLOOK

We have applied a fundamental-physics insight to unitary inversion and quantum sensing. According to decades-old particle physics, one can view positrons as electrons traveling backward in time. We exhibited a positron-like transmon whose effective gyromagnetic ratio equals the negative of an ordinary transmon qubit's. This negativity enabled us to invert a unitary implemented by a magnetic field. We leveraged this platform-specific unitary inversion, with entanglement, in phase estimation: using an entangled qubit-antiquubit pair, we measured the strength of a field pointing in an arbitrary direction. Our strategy works as well even if the direction is unknown. Furthermore, the strategy enabled a greater

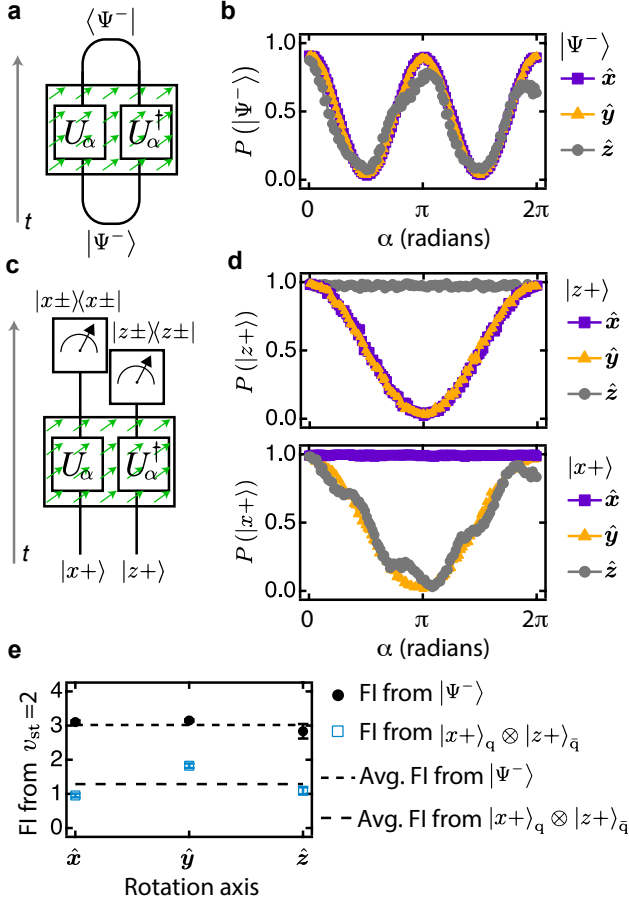


FIG. 3. **Phase estimation with qubit-antiquubit pairs.** (a) Circuit diagram for agnostic phase estimation with synthetic positronium. q and $\bar{q}$ are prepared in the entangled state $|\Psi^-\rangle$, platform-specific unitary inversion effects $U^\dagger$ on the antiquubit, and the pair is measured in the singlet/not-singlet basis. (b) Probability $P(|\Psi^-\rangle)$ that the measurement yields the singlet outcome, plotted against α . Different colors, marker shapes, and line textures correspond to different rotation axes. (c) Circuit diagram for optimal entanglement-free phase estimation performed with a qubit and an antiquubit. (d) Probabilities associated with possible outcomes of the entanglement-free experiment's measurements. (e) FI inferable per $v_{st} = 2$ (i) from positronium metrology and (ii) from phase estimation performed with a separable state of a qubit-antiquubit pair. Error bars result from propagating uncertainty in the curve fitting.

Fisher information, per two units of space-time volume, than two competitor strategies.

This work establishes several opportunities for future research. First, we outline an extension of our scheme to achieve more FI per unit space-time volume. Second, we sketch a use of actual antimatter in quantum sensing. Finally, we identify further applications of platform-specific unitary inversion and synthetic positronium.

By extending our protocol, one can achieve the greatest possible amount of FI per unit space-time volume,

not only the greatest possible amount per two units [45]. Consider subjecting one synthetic-positronium atom to the external field n times sequentially. This protocol achieves an FI of $4n^2$, which equals the QFI. The space-time volume used is $2n$, so the FI per unit space-time volume is $2n$, the optimal value [45].

Our protocol extends not only to n sequential field applications, but also to true antimatter. True positronium offers an advantage over its synthetic counterpart: the positron's gyromagnetic ratio has precisely the same magnitude as the electron's. Hence imperfect control does not threaten the unitary inversion. Moreover, the unitary inversion costs no control resources. One can perform our protocol's measurement [top of Fig. 3(a)] on true positronium using current technology, *positronium annihilation-lifetime spectroscopy* [46]. One measures the time required for a positronium atom to decay into photons. From the time, one can infer the former atom's spin state: the positronium triplets' lifetimes exceed the singlet's lifetime by three orders of magnitude. Yet our protocol's state preparation [bottom of Fig. 3(a)] poses a challenge for positronium, as positronium singlets decay quickly. One can prepare longer-lived entangled triplets, however [47]. Using a positronium triplet, one can achieve an entanglement advantage in a metrological task, albeit not the task in the present paper [48].

True positronium aside, we expect synthetic positronium and platform-specific unitary inversion to find applications beyond metrology. Unitary-inversion methods have enjoyed considerable interest for years, including recently [49–60]. Diverse applications motivate such advances. For instance, efficient unitary inversion underpins quantum algorithms that require black-box access to U_α and $U_\alpha^\dagger$. Example algorithms include amplitude amplification [61], phase estimation [62], and quantum singular-value transformations [13]. Additionally, unitary inversion enables multiparameter sensing that achieves the best possible precision (the Cramér-Rao limit), using a fixed measurement setup [63, 64]. Finally, by efficiently inverting unknown unitaries, an adversary can break certain cryptographic schemes [65]. Hence we expect our platform-specific unitary inversion to enable applications across and beyond metrology.

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ment. X.S. and S.S.B. performed the experiments. S.S.B. and K.M. performed the data analysis. F.S. crafted the theory sections and proofs. N.Y.H., D.R.M.A.S. and K.M. supervised all work; and all authors contributed to the writing and editing of the manuscript.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding authors upon reasonable request.

AUTHOR CONTRIBUTIONS

*Denotes equal contribution. X.S., F.S., Y.-X.W., N.Y.H., D.R.M.A.S. and K.M. conceived of the experi-

COMPETING INTERESTS

The authors declare no competing interests.

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Supplementary Information for “Superconducting antiquubits achieve optimal phase estimation via unitary inversion”

Supplementary Note I concerns the FI achievable using positronium metrology. This FI equals the greatest QFI achievable with any strategy, regardless of whether the field direction is known. Supplementary Note II identifies the greatest possible amount of QFI achievable with synthetic antimatter but without entanglement. This sensing strategy provides a foil for positronium metrology. Supplementary Note III details how we realize antiquubits. In Suppl. Note IV, we describe our experimental setup and quantum gates.

I. FI ACHIEVABLE VIA POSITRONIUM METROLOGY

The main text contains three claims about the FI achievable with positronium metrology: (i) Positronium metrology achieves a FI of $I_\alpha = 4$ while consuming $v_{\text{st}} = 2$ units of space–time volume. (ii) This FI equals the corresponding QFI, which equals the greatest QFI achievable when the field direction is unknown. (iii) In every strategy that satisfies property (ii), one must effectively invert the field experienced by $\bar{q}$. We prove all three claims here.

We begin by introducing terminology. In the main text, *qubit* means *quantum two-level system that acts like ordinary matter* and contrasts with *antiqubit*. When referring to an object that could be a qubit or an antiqubit, we wrote *transmon*. Here, we often refer to a system that (a) could be a qubit or an antiqubit and (b) could be realized with any suitable physical platform, not only with a transmon. We call such an object a *two-level system* (TLS) throughout this supplementary note.

A protocol’s space–time volume equals the product (number of TLSs used) $\times$ (number of sequential unitary applications). Consequently, each of two protocol structures incurs a space–time volume of two. First, one TLS may undergo two consecutive applications of $U_\alpha = e^{-i\alpha H}$. Second, two TLSs may simultaneously undergo U_α or $U_\alpha^\dagger$. We sequentially calculate the QFIs achievable with these protocol structures.

First, consider subjecting one TLS to two sequential unitary applications. Suppose that the rotation axis, $\hat{n}$, is known. The optimal probe state is an equal superposition of the eigenstates of $H = \hat{n} \cdot \boldsymbol{\sigma}/2$, which has eigenvalues $E_\pm = \pm 1/2$. The probe, subjected to two sequential unitaries, undergoes $U_\alpha^2 = e^{-2i\alpha H}$. The effective generator is $2H$, whose spectral gap is 2. By Eq. (2), therefore, the QFI is $\mathcal{I}_\alpha = 4$.

Second, consider two TLSs evolving in parallel, each undergoing U_α or $U_\alpha^\dagger$. In Suppl. Note IA, we derive a general formula for the QFI achievable, in this scenario, with two TLSs prepared in a pure state. We apply the formula to separable states and to maximally entangled states in Suppl. Note IB. Denote the TLSs’ initial joint state by $|\psi\rangle$. Denote its concurrence—a measure of the state’s entanglement [67]—by $C(|\psi\rangle)$. In Suppl. Note IC, we prove that the QFI obeys the upper bound

$$\mathcal{I}_\alpha \leq 2[1 + C(|\psi\rangle)] \leq 4. \quad (6)$$

The concurrence ranges from 0, for separable states, to 1, for maximally entangled states (Bell pairs). Furthermore, pure initial states enable the greatest possible QFI: if the initial state is mixed, quantum information has leaked from the probe to the environment, preventing the metrologist from gathering all possible information from the probe. Hence 4 is the greatest QFI achievable with any initial state of two TLSs that experience a field simultaneously.

To achieve this QFI without knowing the rotation axis, one must effectively invert the field, we prove in Suppl. Note ID. Positronium metrology effectively inverts the field by simulating antimatter. In Suppl. Note IE, we prove that positronium metrology can achieve an FI equal to this QFI. Therefore, positronium metrology can achieve an FI equal to the greatest possible QFI achievable with two units of space–time volume.

A. QFI achievable with two TLSs and one application of the external field

In this supplementary note, we derive a formula for the QFI achievable with two TLSs, each of which undergoes U_α or $U_\alpha^\dagger$. This formula lays the foundation for later supplementary notes: in Suppl. Note IB, we apply the formula to separable states and to maximally entangled states. In Suppl. Note IC, we use the formula to prove the concurrence bound (6). We use the formula, in Suppl. Note ID, to demonstrate that unitary inversion and entanglement enable the unique strategy whose FI can achieve the greatest QFI attainable if one does not know the field’s direction.

Consider rotating a TLS through an unknown angle α about the axis $\hat{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$. The unitary $U_\alpha = e^{-i\alpha\hat{n}\cdot\boldsymbol{\sigma}/2}$ represents this operation. Consider subjecting one TLS to U_α and another TLS to either

U_α or $U_\alpha^\dagger$. The TLS pair's joint state evolves under

$$\mathcal{U}_\alpha^{(+)} := U_\alpha \otimes U_\alpha = e^{-i\alpha \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2} \otimes e^{-i\alpha \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2} = e^{-i\alpha (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma})/2} \quad \text{or} \quad (7)$$

$$\mathcal{U}_\alpha^{(-)} := U_\alpha \otimes U_\alpha^\dagger = e^{-i\alpha \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2} \otimes e^{+i\alpha \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2} = e^{-i\alpha (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} - \mathbb{1} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma})/2}. \quad (8)$$

We synopsise both equations in

$$\mathcal{U}_\alpha^{(\pm)} = e^{-i\alpha (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} \pm \mathbb{1} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma})/2} =: e^{-i\alpha \mathcal{H}^{(\pm)}/2}. \quad (9)$$

The effective Hamiltonian has the form $\mathcal{H}^{(\pm)} := \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} \pm \mathbb{1} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$.

Let us calculate the QFI one can achieve by evolving a two-TLS state $|\psi\rangle$ under $\mathcal{U}_\alpha^{(\pm)}$. The following calculation does not depend on whether an agent prepared $|\psi\rangle$ using knowledge of $\hat{\mathbf{n}}$.

First, we introduce notation. We label the first TLS as A and the second as B . We define a tensor T in terms of its elements,

$$T_{ij} := \langle \psi | \sigma_i \otimes \sigma_j | \psi \rangle. \quad (10)$$

The TLSs initially have the Bloch vectors

$$\mathbf{r}^{(A)} := \langle \psi | \boldsymbol{\sigma} \otimes \mathbb{1} | \psi \rangle \quad \text{and} \quad \mathbf{r}^{(B)} := \langle \psi | \mathbb{1} \otimes \boldsymbol{\sigma} | \psi \rangle. \quad (11)$$

Projecting the Bloch vectors onto $\hat{\mathbf{n}}$ yields

$$r_{\hat{\mathbf{n}}}^{(i)} := \hat{\mathbf{n}} \cdot \mathbf{r}^{(i)}, \quad \text{wherein} \quad i = A, B. \quad (12)$$

Finally, let $s := \pm 1$. In terms of the quantities above, we can express the QFI achievable with the probe prepared in $|\psi\rangle$. The QFI equals the variance of $\mathcal{H}^{(\pm)}$, calculated in the rest of this supplementary note:

$$\mathcal{I}_\alpha^{(s)} = \left(\Delta \mathcal{H}^{(s)} \right)^2 = 2 \left(1 + s \hat{\mathbf{n}}^T T \hat{\mathbf{n}} \right) - \left(r_{\hat{\mathbf{n}}}^{(A)} + s r_{\hat{\mathbf{n}}}^{(B)} \right)^2. \quad (13)$$

Let us prove Eq. (13). The effective Hamiltonian has a variance

$$\left(\Delta \mathcal{H}^{(\pm)} \right)^2 = \langle \psi | \left(\mathcal{H}^{(\pm)} \right)^2 | \psi \rangle - \langle \psi | \mathcal{H}^{(\pm)} | \psi \rangle^2. \quad (14)$$

We calculate the two terms sequentially. The first has the form

$$\langle \psi | \left(\mathcal{H}^{(\pm)} \right)^2 | \psi \rangle = \langle \psi | (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} \pm \mathbb{1} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma})^2 | \psi \rangle \quad (15)$$

$$= \langle \psi | (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})^2 \otimes \mathbb{1} + \mathbb{1} \otimes (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})^2 \pm 2 (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) | \psi \rangle \quad (16)$$

$$= \langle \psi | (\mathbb{1} \otimes \mathbb{1}) + (\mathbb{1} \otimes \mathbb{1}) \pm 2 (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) | \psi \rangle \quad (17)$$

$$= 2 [1 \pm \langle \psi | (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) | \psi \rangle] \quad (18)$$

$$= 2 [1 \pm \hat{\mathbf{n}} \cdot \langle \psi | (\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}) | \psi \rangle \cdot \hat{\mathbf{n}}] \quad (19)$$

$$= 2 (1 \pm \hat{\mathbf{n}}^T T \hat{\mathbf{n}}). \quad (20)$$

The second term in Eq. (14) has the form

$$\langle \psi | \mathcal{H}^{(\pm)} | \psi \rangle^2 = \langle \psi | \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} \pm \mathbb{1} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} | \psi \rangle^2 \quad (21)$$

$$= (\langle \psi | \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} | \psi \rangle \pm \langle \psi | \mathbb{1} \otimes \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} | \psi \rangle)^2 \quad (22)$$

$$= (\hat{\mathbf{n}} \cdot \langle \psi | \boldsymbol{\sigma} \otimes \mathbb{1} | \psi \rangle \pm \hat{\mathbf{n}} \cdot \langle \psi | \mathbb{1} \otimes \boldsymbol{\sigma} | \psi \rangle)^2 \quad (23)$$

$$= \left(\hat{\mathbf{n}} \cdot \mathbf{r}^{(A)} \pm \hat{\mathbf{n}} \cdot \mathbf{r}^{(B)} \right)^2 \quad (24)$$

$$= \left(r_{\hat{\mathbf{n}}}^{(A)} \pm r_{\hat{\mathbf{n}}}^{(B)} \right)^2. \quad (25)$$

Inserting Eqs. (20) and (25) into Eq. (14) yields Eq. (13), the formula we aimed to prove.

B. QFI achievable with two TLSs prepared in a maximally entangled state or in a separable state

In this supplementary note, we apply Eq. (13) to two cases. First, we calculate the QFI achievable with two TLSs prepared in a maximally entangled state, such as the singlet state used in positronium metrology. Then, we calculate the QFI achievable with a separable state. This analysis illuminates how the QFI depends on the initial state's entanglement.

Maximally entangled states: Suppose the TLSs are prepared in a maximally entangled $|\psi\rangle$. Each TLS's reduced state is maximally mixed: $\text{Tr}_B(|\psi\rangle\langle\psi|) = \frac{1}{2}\mathbb{1}_A$, and $\text{Tr}_A(|\psi\rangle\langle\psi|) = \frac{1}{2}\mathbb{1}_B$. Hence, since the Pauli operators are traceless, $\mathbf{r}^{(A)} = \frac{1}{2}\text{Tr}(\boldsymbol{\sigma} \otimes \mathbb{1}) = \frac{1}{2}\text{Tr}(\boldsymbol{\sigma}) = \mathbf{0}$. The Bloch vector $\mathbf{r}^{(B)}$ vanishes analogously. Hence the Bloch vectors' projections vanish, $\mathbf{r}_{\hat{\mathbf{n}}}^{(A)} = \mathbf{r}_{\hat{\mathbf{n}}}^{(B)} = \mathbf{0}$, regardless of $\hat{\mathbf{n}}$. Therefore, Eq. (13) reduces to

$$\mathcal{I}_{\alpha}^{(s)} = 2(1 + s \hat{\mathbf{n}}^T T \hat{\mathbf{n}}). \quad (26)$$

In positronium metrology, the two-TLS state $|\psi\rangle = |\Psi^-\rangle$ is the singlet. The qubit probe and antiqubit ancilla rotate oppositely, so $s = -1$. This state's $T = -\mathbb{1}$, so Eq. (26) simplifies to

$$\mathcal{I}_{\alpha}^{(-)} = 2[1 - \hat{\mathbf{n}}^T (-\mathbb{1}) \hat{\mathbf{n}}] = 2(1 + |\hat{\mathbf{n}}|^2) = 4. \quad (27)$$

Hence the QFI does not depend on the rotation axis.

Suppose that $|\psi\rangle$ is any other maximally entangled state. T is a diagonal matrix with a determinant of 1. However, T is not proportional to the identity matrix, as it does if $|\psi\rangle$ is a singlet. Consequently, the QFI depends on the rotation axis. $\mathcal{I}_{\alpha}^{(s)}$ achieves its maximum value (which equals 4, we show shortly) when the rotation axis is an eigenvalue- s eigenvector of T : $T\hat{\mathbf{n}} = s\hat{\mathbf{n}}$. Under this condition, the QFI evaluates to

$$\mathcal{I}_{\alpha}^{(s)} = 2(1 + s \hat{\mathbf{n}}^T T \hat{\mathbf{n}}) = 2(1 + s \hat{\mathbf{n}}^T s \hat{\mathbf{n}}) = 2(1 + s^2 \hat{\mathbf{n}}^2) = 4. \quad (28)$$

In Suppl. Note IC, we identify the states $|\psi\rangle$ that satisfy the maximum-QFI condition $T\hat{\mathbf{n}} = s\hat{\mathbf{n}}$.

Separable states: Now, suppose that the two TLSs are prepared in a separable $|\psi\rangle$. By Eqs. (10) and (11), $T = \mathbf{r}^{(A)}\mathbf{r}^{(B)T}$. We substitute the right-hand side into the QFI formula (13):

$$\mathcal{I}_{\alpha}^{(s)} = 2\left(1 + s \hat{\mathbf{n}}^T \mathbf{r}^{(A)}\mathbf{r}^{(B)T} \hat{\mathbf{n}}\right) - \left(r_{\hat{\mathbf{n}}}^{(A)} + s r_{\hat{\mathbf{n}}}^{(B)}\right)^2 = 2\left(1 + s r_{\hat{\mathbf{n}}}^{(A)} r_{\hat{\mathbf{n}}}^{(B)}\right) - \left(r_{\hat{\mathbf{n}}}^{(A)} + s r_{\hat{\mathbf{n}}}^{(B)}\right)^2 \quad (29)$$

$$= 2\left[1 - \left(r_{\hat{\mathbf{n}}}^{(A)2} + r_{\hat{\mathbf{n}}}^{(B)2}\right)\right] \leq 2. \quad (30)$$

This QFI maximizes when $r_{\hat{\mathbf{n}}}^{(A)} = r_{\hat{\mathbf{n}}}^{(B)} = 0$: both TLSs' initial Bloch vectors point orthogonally to $\hat{\mathbf{n}}$.

C. Proof of the bound (6) on the QFI achievable with two TLSs

In this supplementary note, we bound the QFI $\mathcal{I}_{\alpha}^{(s)}$ in terms of the two TLSs' initial concurrence, $C(|\psi\rangle)$. We reproduce the main result, Eq. (6), here for convenience:

$$\mathcal{I}_{\alpha}^{(s)} \leq 2[1 + C(\psi)]. \quad (31)$$

The concurrence C quantifies entanglement [67], ranging from 0 (for separable states) to 1 (for maximally entangled states). We also identify the two-TLS states that saturate Eq. (31).

Our strategy is as follows. First, we consider a general two-TLS state $|\psi\rangle$ that has a fixed concurrence $C(|\psi\rangle) = C_0$. We parameterize $|\psi\rangle$ using local unitaries acting on a simpler reference state $|\chi\rangle$ that has the concurrence C_0 . This parameterization leverages the concurrence's invariance under local unitary operations. We compute the reference state's QFI, using Eq. (13). To simplify the analysis, we move to a rotated frame, in which the QFI depends only on the relative rotation between the TLSs, $U_{\text{rel}} := U_A^{-1}U_B$. Finally, we reformulate the QFI as a quadratic form that reveals the optimal relative rotation's structure. This reformulation yields the bound (31), as well as a complete characterization of the states $|\psi\rangle$ that saturate it.

1. General pure two-TLS state and concurrence

A general two-TLS pure state has the form

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \quad \text{wherein} \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1. \quad (32)$$

This state has the concurrence [68]

$$C_0 := C(|\psi\rangle) = |ad - bc| \in [0, 1]. \quad (33)$$

Denote by $|\chi\rangle$ a reference state that has the same concurrence, $C(|\chi\rangle) = C_0$:

$$|\chi\rangle := \sqrt{\lambda_1} |00\rangle + \sqrt{\lambda_2} |11\rangle, \quad (34)$$

wherein $\lambda_{1,2} = \frac{1 \pm \sqrt{1 - C_0^2}}{2}$. From $|\chi\rangle$, one can generate every concurrence- C_0 two-TLS state $|\psi\rangle$ via single-TLS unitaries U_A and U_B [69]:

$$|\psi\rangle = (U_A \otimes U_B) |\chi\rangle. \quad (35)$$

2. QFI achievable with fixed-concurrence states

In this section, we compute the QFI achievable with a general pure two-TLS state $|\psi\rangle$ of fixed concurrence C_0 . Directly evaluating the QFI is algebraically cumbersome. Yet we can simplify the calculation because every such state decomposes as $|\psi\rangle = (U_A \otimes U_B) |\chi\rangle$ in terms of the reference state $|\chi\rangle$ introduced in Eq. (34). Evaluating the QFI achievable with $|\chi\rangle$ is simple: the state's correlation tensor T is diagonal, and its Bloch vectors $\mathbf{r}^{(A)}, \mathbf{r}^{(B)} = \hat{\mathbf{z}}$. We incorporate the local unitaries' effects into the QFI using the unitaries, R_A and R_B , that represent the corresponding rotations on the Bloch sphere. A simple QFI expression results; we optimize it across the field directions $\hat{\mathbf{n}}$ and across the concurrence- C_0 states.

To calculate the QFI of $|\psi\rangle$, we use (13):

$$\mathcal{I}_\alpha^{(s)} = \left(\Delta \mathcal{H}^{(s)} \right)^2 = 2 \left(1 + s \hat{\mathbf{n}}^T T \hat{\mathbf{n}} \right) - \left(r_{\hat{\mathbf{n}}}^{(A)} + s r_{\hat{\mathbf{n}}}^{(B)} \right)^2. \quad (36)$$

First, we compute T , $r_{\hat{\mathbf{n}}}^{(A)}$, and $r_{\hat{\mathbf{n}}}^{(B)}$:

$$T_{ij} := \langle \psi | \sigma_i \otimes \sigma_j | \psi \rangle = \langle \chi | \left(U_A^\dagger \otimes U_B^\dagger \right) \sigma_i \otimes \sigma_j (U_A \otimes U_B) | \chi \rangle = \langle \chi | U_A^\dagger \sigma_i U_A \otimes U_B^\dagger \sigma_j U_B | \chi \rangle \quad (37)$$

$$= (R_A)_{ik} (R_B)_{j\ell} \langle \chi | \sigma_k \otimes \sigma_\ell | \chi \rangle. \quad (38)$$

The matrices $R_A, R_B \in \text{SO}(3)$ describe how the local unitaries U_A and U_B rotate the Pauli vectors $\sigma_{i,j}$ across the Bloch sphere:

$$U_X^\dagger \sigma_i U_X = (R_X)_{ij} \sigma_j, \quad \text{wherein} \quad X = A, B. \quad (39)$$

Directly calculating the final factor in Eq. (38), $\langle \chi | \sigma_k \otimes \sigma_\ell | \chi \rangle$, yields

$$T = R_A \begin{pmatrix} C_0 & 0 & 0 \\ 0 & -C_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_B^T \equiv R_A D R_B^T. \quad (40)$$

We have defined the diagonal matrix $D := \text{diag}(C_0, -C_0, 1)$. Having calculated the T in Eq. (36), we compute the Bloch-vector projections $r_{\hat{\mathbf{n}}}^{(A)} := \hat{\mathbf{n}} \cdot \mathbf{r}^{(A)}$ and $r_{\hat{\mathbf{n}}}^{(B)} := \hat{\mathbf{n}} \cdot \mathbf{r}^{(B)}$. We can express the i^{th} component of $\mathbf{r}^{(A)}$ in terms of the rotation matrix R_A and the concurrence C_0 :

$$\mathbf{r}^{(A)}_i := \langle \psi | \sigma_i \otimes \mathbb{1} | \psi \rangle = \langle \chi | U_A^\dagger \sigma_i U_A \otimes \mathbb{1} | \chi \rangle = (R_A)_{ik} \langle \chi | \sigma_k \otimes \mathbb{1} | \chi \rangle = \sqrt{1 - C_0^2} (R_A \hat{\mathbf{z}})_i. \quad (41)$$

$\mathbf{r}^{(B)}$ obeys an analogous equation. In summary,

$$\mathbf{r}^{(A)} = \sqrt{1 - C_0^2} R_A \hat{\mathbf{z}}, \quad \text{and} \quad (42)$$

$$\mathbf{r}^{(B)} = \sqrt{1 - C_0^2} R_B \hat{\mathbf{z}}. \quad (43)$$

Let us substitute the calculated quantities from Eqs. (40), (42) and (43) into the QFI formula (13):

$$\mathcal{I}_\alpha^{(s)} = 2 \left(1 + s \hat{\mathbf{n}}^T T \hat{\mathbf{n}} \right) - \left(r_{\hat{\mathbf{n}}}^{(A)} + s r_{\hat{\mathbf{n}}}^{(B)} \right)^2 = 2 + 2s \hat{\mathbf{n}}^T T \hat{\mathbf{n}} - \left[\hat{\mathbf{n}} \cdot \left(\mathbf{r}^{(A)} + s \mathbf{r}^{(B)} \right) \right]^2 \quad (44)$$

$$= 2 + 2s \hat{\mathbf{n}}^T (R_A D R_B^T) \hat{\mathbf{n}} - (1 - C_0^2) \left[\hat{\mathbf{n}}^T (R_A \hat{\mathbf{z}} + s R_B \hat{\mathbf{z}}) \right]^2. \quad (45)$$

3. QFI achievable with a general two-TLS pure state, calculated in a rotated reference frame

Equation (45) shows a QFI dependent on the Bloch-sphere rotations $R_A, R_B \in \text{SO}(3)$. They represent the single-TLS unitaries U_A and U_B . We now analyze how these rotations affect the QFI. According to Eq. (35), every pure, concurrence- C_0 two-TLS state $|\psi\rangle$ results from applying single-TLS unitaries to a concurrence- C_0 reference state $|\chi\rangle$. Those unitaries decompose into (i) two identical single-TLS unitaries U_{id} and (ii) a relative rotation U_{rel} :

$$|\psi\rangle = (U_{\text{id}} \otimes U_{\text{id}})(\mathbb{1} \otimes U_{\text{rel}})|\chi\rangle. \quad (46)$$

We now show that the greatest possible QFI (the maximum over field directions $\hat{\mathbf{n}}$) is invariant under identical rotations of both TLSs. Let $S \in \text{SO}(3)$ denote an arbitrary rotation matrix: $SS^T = S^T S = \mathbb{1}$. We define a rotated reference frame by applying S to the original rotation matrices (R_A and R_B) and to the field direction ($\hat{\mathbf{n}}$):

$$R_A \mapsto S R_A, \quad R_B \mapsto S R_B, \quad \text{and} \quad \hat{\mathbf{n}} \mapsto S \hat{\mathbf{n}}. \quad (47)$$

The QFI is invariant under this rotation. We prove this claim by transforming each component of Eq. (45) that is not overtly a scalar:

$$\hat{\mathbf{n}}^T (R_A D R_B^T) \hat{\mathbf{n}} \mapsto (S \hat{\mathbf{n}})^T [S R_A D (S R_B)^T] (S \hat{\mathbf{n}}) = \hat{\mathbf{n}}^T S^T S (R_A D R_B^T) S^T S \hat{\mathbf{n}} = \hat{\mathbf{n}}^T (R_A D R_B^T) \hat{\mathbf{n}}, \text{ and} \quad (48)$$

$$\hat{\mathbf{n}}^T (R_A \hat{\mathbf{z}} + s R_B \hat{\mathbf{z}}) \mapsto (S \hat{\mathbf{n}})^T (S R_A \hat{\mathbf{z}} + s S R_B \hat{\mathbf{z}}) = \hat{\mathbf{n}}^T S^T S (R_A \hat{\mathbf{z}} + s R_B \hat{\mathbf{z}}) = \hat{\mathbf{n}}^T (R_A \hat{\mathbf{z}} + s R_B \hat{\mathbf{z}}). \quad (49)$$

Without loss of generality, we select $S = R_A^{-1}$. By the mapping in Eq. (47), $R_A \mapsto S R_A = \mathbb{1}$, and $R_B \mapsto S R_B = R_A^{-1} R_B =: R_{\text{rel}}$. To calculate the effect of S on the QFI, we apply these mappings to Eq. (45):

$$\mathcal{I}_\alpha^{(s)} = 2 + 2s (S \hat{\mathbf{n}})^T \left[(S R_A) D (S R_B)^T \right] (S \hat{\mathbf{n}}) - (1 - C_0^2) \left\{ (S \hat{\mathbf{n}})^T [(S R_A) \hat{\mathbf{z}} + s (S R_B) \hat{\mathbf{z}}] \right\}^2 \quad (50)$$

$$= 2 + 2 \left\{ s (S \hat{\mathbf{n}})^T (D R_{\text{rel}}^T) (S \hat{\mathbf{n}}) - (1 - C_0^2) \left[(S \hat{\mathbf{n}})^T (\hat{\mathbf{z}} + s R_{\text{rel}} \hat{\mathbf{z}}) \right]^2 \right\} \quad (51)$$

$$\leq 2 + 2 \max_{\hat{\mathbf{n}}} \left\{ s \hat{\mathbf{n}}^T (D R_{\text{rel}}^T) \hat{\mathbf{n}} - (1 - C_0^2) \left[\hat{\mathbf{n}}^T (\hat{\mathbf{z}} + s R_{\text{rel}} \hat{\mathbf{z}}) \right]^2 \right\}. \quad (52)$$

Equation (51) reveals that rotating the reference frame, using S , amounts to redefining $\hat{\mathbf{n}}$. By maximizing over $\hat{\mathbf{n}}$ in Eq. (52), we removed the S -dependence, reformulating Eq. (45) in a way that only depends on the relative rotation R_{rel} .

4. Greatest QFI achievable with any fixed-concurrence two-TLS state

In this section, we identify the external-field directions $\hat{\mathbf{n}}$ and the relative rotation R_{rel} that maximize the QFI achievable with a concurrence- C_0 state. One might hope to maximize the first term inside the curly braces in Eq. (52), while keeping the second (nonpositive) term zero. This strategy is unfortunately impossible, except when $C_0 = 1$. We thus rewrite Eq. (52) in a form that points to the optimal relative rotation R_{rel} .

We rewrite the bound as follows. First, we expand the term proportional to $(1 - C_0^2)$. Then, we express the diagonal

matrix as D as $\text{diag}(C_0, -C_0, 1) = C_0 \hat{\mathbf{x}} \hat{\mathbf{x}}^T - C_0 \hat{\mathbf{y}} \hat{\mathbf{y}}^T + \hat{\mathbf{z}} \hat{\mathbf{z}}^T$. The bound assumes the form

$$\mathcal{I}_\alpha^{(s)} \leq 2 + 2 \max_{\hat{\mathbf{n}}} \left\{ s \hat{\mathbf{n}}^T (D R_{\text{rel}}^T) \hat{\mathbf{n}} - (1 - C_0^2) [\hat{\mathbf{n}}^T (\hat{\mathbf{z}} + s R_{\text{rel}} \hat{\mathbf{z}})]^2 \right\} \quad (53)$$

$$= 2 + 2 \max_{\hat{\mathbf{n}}} \left\{ s \hat{\mathbf{n}}^T (D R_{\text{rel}}^T) \hat{\mathbf{n}} - (1 - C_0^2) \hat{\mathbf{n}}^T (\hat{\mathbf{z}} + s R_{\text{rel}} \hat{\mathbf{z}}) (\hat{\mathbf{z}} + s R_{\text{rel}} \hat{\mathbf{z}})^T \hat{\mathbf{n}} \right\} \quad (54)$$

$$= 2 + 2 \max_{\hat{\mathbf{n}}} \hat{\mathbf{n}}^T \left\{ s (\hat{\mathbf{z}} \hat{\mathbf{z}}^T R_{\text{rel}}^T) + s C_0 (\hat{\mathbf{x}} \hat{\mathbf{x}}^T - \hat{\mathbf{y}} \hat{\mathbf{y}}^T) R_{\text{rel}}^T \right. \\ \left. - (1 - C_0^2) [\hat{\mathbf{z}} \hat{\mathbf{z}}^T + s \hat{\mathbf{z}} (R_{\text{rel}} \hat{\mathbf{z}})^T + s (R_{\text{rel}} \hat{\mathbf{z}}) \hat{\mathbf{z}}^T + (R_{\text{rel}} \hat{\mathbf{z}})(R_{\text{rel}} \hat{\mathbf{z}})^T] \right\} \hat{\mathbf{n}} \quad (55)$$

$$= 2 + 2 \max_{\hat{\mathbf{n}}} \hat{\mathbf{n}}^T \left\{ s C_0^2 (\hat{\mathbf{z}} \hat{\mathbf{z}}^T R_{\text{rel}}^T) + s C_0 (\hat{\mathbf{x}} \hat{\mathbf{x}}^T - \hat{\mathbf{y}} \hat{\mathbf{y}}^T) R_{\text{rel}}^T \right. \\ \left. - (1 - C_0^2) [\hat{\mathbf{z}} \hat{\mathbf{z}}^T + s (R_{\text{rel}} \hat{\mathbf{z}}) \hat{\mathbf{z}}^T + (R_{\text{rel}} \hat{\mathbf{z}})(R_{\text{rel}} \hat{\mathbf{z}})^T] \right\} \hat{\mathbf{n}} \quad (56)$$

$$= 2 + 2 \max_{\hat{\mathbf{n}}} \hat{\mathbf{n}}^T \left\{ s C_0 (\hat{\mathbf{x}} \hat{\mathbf{x}}^T - \hat{\mathbf{y}} \hat{\mathbf{y}}^T + C_0 \hat{\mathbf{z}} \hat{\mathbf{z}}^T) R_{\text{rel}}^T - (1 - C_0^2) [\hat{\mathbf{z}} \hat{\mathbf{z}}^T + s (R_{\text{rel}} \hat{\mathbf{z}}) \hat{\mathbf{z}}^T + (R_{\text{rel}} \hat{\mathbf{z}})(R_{\text{rel}} \hat{\mathbf{z}})^T] \right\} \hat{\mathbf{n}}. \quad (57)$$

The term proportional to $(1 - C_0^2)$ yields a nonpositive contribution to the QFI, due to two facts. First, $-(1 - C_0^2)$ is nonpositive for all $C_0 \in [0, 1]$. Second, the quadratic form is non-negative for all $\hat{\mathbf{n}}$:

$$\hat{\mathbf{n}}^T [\hat{\mathbf{z}} \hat{\mathbf{z}}^T + s (R_{\text{rel}} \hat{\mathbf{z}}) \hat{\mathbf{z}}^T + (R_{\text{rel}} \hat{\mathbf{z}})(R_{\text{rel}} \hat{\mathbf{z}})^T] \hat{\mathbf{n}} = |\hat{\mathbf{n}}^T \hat{\mathbf{z}}|^2 + |\hat{\mathbf{n}}^T R_{\text{rel}} \hat{\mathbf{z}}|^2 + s (\hat{\mathbf{n}}^T R_{\text{rel}} \hat{\mathbf{z}}) (\hat{\mathbf{z}}^T \hat{\mathbf{n}}) \quad (58)$$

$$= \left(\hat{\mathbf{n}}^T \hat{\mathbf{z}} + \frac{s}{2} \hat{\mathbf{n}}^T R_{\text{rel}} \hat{\mathbf{z}} \right)^2 + \left(1 - \frac{s^2}{4} \right) |\hat{\mathbf{n}}^T R_{\text{rel}} \hat{\mathbf{z}}|^2 \geq 0. \quad (59)$$

This inequality saturates only if both terms in Eq. (59) vanish. The second term vanishes when $\hat{\mathbf{n}}^T R_{\text{rel}} \hat{\mathbf{z}} = 0$, such that $\hat{\mathbf{n}} \perp R_{\text{rel}} \hat{\mathbf{z}}$. Under this condition, the first term in Eq. (59) vanishes when $\hat{\mathbf{n}}^T \hat{\mathbf{z}} = 0$, such that $\hat{\mathbf{n}} \perp \hat{\mathbf{z}}$. In summary, the inequality (59) saturates if and only if $\hat{\mathbf{n}} \perp \hat{\mathbf{z}}, R_{\text{rel}} \hat{\mathbf{z}}$. This condition is satisfied, for example, when $\hat{\mathbf{n}} \perp \hat{\mathbf{z}}$ and $R_{\text{rel}} \hat{\mathbf{n}} = \pm \hat{\mathbf{n}}$.

Having identified under which condition the nonpositive contribution to the QFI vanishes, we demonstrate that this condition maximizes the term proportional to $s C_0$ in Eq. (57). The analysis decomposes into two distinct cases:

- $C_0 \neq 1$: The contribution proportional to $s C_0$ in Eq. (57), $s C_0 \hat{\mathbf{n}}^T (\hat{\mathbf{x}} \hat{\mathbf{x}}^T - \hat{\mathbf{y}} \hat{\mathbf{y}}^T + C_0 \hat{\mathbf{z}} \hat{\mathbf{z}}^T) R_{\text{rel}}^T \hat{\mathbf{n}}$, assumes its maximum, $+1$, if the field direction $\hat{\mathbf{n}}$ is an eigenvector, associated with the eigenvalue $s' := \pm 1$, of R_{rel}^T : $R_{\text{rel}}^T \hat{\mathbf{n}} = s' \hat{\mathbf{n}}$. If $ss' = -1$, then $\hat{\mathbf{n}} = \hat{\mathbf{y}}$. If $ss' = 1$, then $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. Under each of these two conditions, the term proportional to $(1 - C_0^2)$ in Eq. (57) vanishes. The QFI simplifies to

$$\mathcal{I}_\alpha^{(s)} = 2(1 + C_0). \quad (60)$$

Every other R_{rel} or $\hat{\mathbf{n}}$ leads to a smaller QFI: the term proportional to $s C_0$ in Eq. (57) shrinks, and the term proportional to $(1 - C_0^2)$ contributes a negative value.

- $C_0 = 1$: The term proportional to $(1 - C_0^2)$ in Eq. (57) vanishes identically. The maximal QFI is therefore determined entirely by the contribution proportional to $s C_0$, $s C_0 \hat{\mathbf{n}}^T (\hat{\mathbf{x}} \hat{\mathbf{x}}^T - \hat{\mathbf{y}} \hat{\mathbf{y}}^T + \hat{\mathbf{z}} \hat{\mathbf{z}}^T) R_{\text{rel}}^T \hat{\mathbf{n}}$. This expression assumes its maximum, $+1$, when $R_{\text{rel}}^T \hat{\mathbf{n}} = s' \hat{\mathbf{n}}$. If $ss' = -1$, then $\hat{\mathbf{n}} = \hat{\mathbf{y}}$. If $ss' = +1$, then $\hat{\mathbf{n}}$ lies in the $\hat{\mathbf{x}} \hat{\mathbf{z}}$ -plane. Under each of these two conditions, the QFI achieves its maximal value, $\mathcal{I}_\alpha^{(s)} = 4$. The QFI maximizes, for example, if $s = -1$ and $R_{\text{rel}}^T = \text{diag}(-1, 1, -1)$. This R_{rel} represents a π rotation about the y -axis. The achievable QFI is optimal in this case, regardless of $\hat{\mathbf{n}}$.

In summary, we have shown that the greatest QFI achievable with a concurrence- C_0 input state $|\psi\rangle$ is $\mathcal{I}_\alpha^{(s)} = 2(1 + C_0)$, if $C_0 < 1$, or $\mathcal{I}_\alpha^{(s)} = 4$, if $C_0 = 1$. We have therefore proved Eq. (31).

5. Optimal input states

We now characterize the optimal two-TLS input states consistent with a fixed concurrence C_0 . We define the optimal input states $|\psi\rangle$ as those that, while having a concurrence C_0 , saturate the upper bound (31). They achieve the greatest possible QFI, $\mathcal{I}_\alpha^{(s)} = 2(1 + C_0)$.

We pursue the following strategy. The previous supplementary note specifies the relative rotation matrix R_{rel} that maximizes the QFI. We construct the single-TLS unitary U_{rel} that implements this R_{rel} . By substituting the unitary's form into Eq. (46), we construct the optimal input states. We enumerate all the possible cases, distinguished by (i) the concurrence's value ($C_0 \neq 1$ or $C_0 = 1$) and (ii) whether the TLSs rotate in the same direction ($s = \pm 1$). Then, we show that positronium metrology exemplifies one of these cases and so achieves the greatest possible QFI. Throughout this supplementary note, U_{id} denotes an arbitrary single-TLS unitary, representing the collective rotation introduced in Eq. (46).

Recall that, when $C_0 \neq 1$, the QFI is maximized when $R_{\text{rel}}^T \hat{\mathbf{n}} = s' \hat{\mathbf{n}}$, wherein $\hat{\mathbf{n}} = \hat{\mathbf{y}}$ (if $ss' = -1$) or $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ (if $ss' = 1$). Hence, if $C_0 \neq 1$ and $s = -1$, the QFI is maximized when $R_{\text{rel}} \hat{\mathbf{y}} = \hat{\mathbf{y}}$ or $R_{\text{rel}} \hat{\mathbf{x}} = -\hat{\mathbf{x}}$. First, suppose that $R_{\text{rel}} \hat{\mathbf{y}} = \hat{\mathbf{y}}$: the relative rotation is through an arbitrary angle ϕ about the y -axis. The unitary $U_{\text{rel}} = \exp\left(-i\frac{\phi}{2} \sigma_y\right)$ implements this rotation. Now, suppose that $R_{\text{rel}} \hat{\mathbf{x}} = -\hat{\mathbf{x}}$: the relative rotation is through an angle π about an axis perpendicular to $\hat{\mathbf{x}}$. The unitary $U_{\text{rel}} = \exp\left(-i\frac{\pi}{2} [\cos(\phi) \sigma_y + \sin(\phi) \sigma_z]\right)$ implements this rotation. The rotation axis, perpendicular to $\hat{\mathbf{x}}$, lies in the yz -plane, at an arbitrary angle ϕ to the $\hat{\mathbf{y}}$ -axis. By Eq. (46), the QFI is maximized if

$$|\psi\rangle = \begin{cases} \left[U_{\text{id}} \otimes U_{\text{id}} \exp\left(-i\frac{\phi}{2} \sigma_y\right)\right] |\chi\rangle & \text{if } s = -1. \\ \left\{U_{\text{id}} \otimes U_{\text{id}} \exp\left(-i\frac{\pi}{2} [\cos(\phi) \sigma_y + \sin(\phi) \sigma_z]\right)\right\} |\chi\rangle, & \end{cases} \quad (61)$$

Now, suppose that $C_0 \neq 1$ and $s = 1$. The QFI is maximized when $R_{\text{rel}} \hat{\mathbf{x}} = \hat{\mathbf{x}}$ or $R_{\text{rel}} \hat{\mathbf{y}} = -\hat{\mathbf{y}}$. We now identify the unitary that implements each of these possible relative rotations. If $R_{\text{rel}} \hat{\mathbf{x}} = \hat{\mathbf{x}}$, then $U_{\text{rel}} = \exp\left(-i\frac{\phi}{2} \sigma_x\right)$. If $R_{\text{rel}} \hat{\mathbf{y}} = -\hat{\mathbf{y}}$, then $U_{\text{rel}} = \exp\left(-i\frac{\pi}{2} [\cos(\phi) \sigma_x + \sin(\phi) \sigma_z]\right)$. By Eq. (46), the optimal input state has the form

$$|\psi\rangle = \begin{cases} \left[U_{\text{id}} \otimes U_{\text{id}} \exp\left(-i\frac{\phi}{2} \sigma_x\right)\right] |\chi\rangle & \text{if } s = 1. \\ \left\{U_{\text{id}} \otimes U_{\text{id}} \exp\left(-i\frac{\pi}{2} [\cos(\phi) \sigma_x + \sin(\phi) \sigma_z]\right)\right\} |\chi\rangle, & \end{cases} \quad (62)$$

Finally, suppose that $C_0 = 1$. We identify similarly the states that saturate the QFI bound Eq. (31):

$$|\psi\rangle = \begin{cases} \left[U_{\text{id}} \otimes U_{\text{id}} \exp\left(-i\frac{\phi}{2} \sigma_y\right)\right] |\chi\rangle & \text{if } s = -1, \\ (U_{\text{id}} \otimes -iU_{\text{id}} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) |\chi\rangle, & \end{cases} \quad (63)$$

and

$$|\psi\rangle = \left\{U_{\text{id}} \otimes U_{\text{id}} \exp\left(-i\frac{\theta}{2} [\cos(\phi) \sigma_x + \sin(\phi) \sigma_z]\right)\right\} |\chi\rangle, \quad \text{if } s = 1. \quad (64)$$

We now show that positronium metrology exemplifies one of the optimal-QFI conditions above. When $C_0 = 1$ and $s = -1$ [Eq. (63)], one can achieve the greatest possible QFI (under the concurrence constraint), regardless of $\hat{\mathbf{n}}$, with the relative rotation $R_{\text{rel}}^T = \text{diag}(-1, 1, -1)$, a π rotation about the y -axis. The unitary $U_{\text{rel}} = \exp\left(-i\frac{\pi}{2} \sigma_y\right) = -i\sigma_y$ implements this rotation. Since $C_0 = 1$, $|\chi\rangle = |\Phi^+\rangle$. According to Eq. (46), the relative rotation transforms $|\Phi^+\rangle$ into the singlet: $(\mathbb{1} \otimes -i\sigma_y) |\Phi^+\rangle = |\Psi^-\rangle$. The singlet is invariant under tensor products of identical rotations: $(U_{\text{id}} \otimes U_{\text{id}}) |\Psi^-\rangle = e^{i\gamma} |\Psi^-\rangle$, wherein γ denotes a global-phase angle. Thus, the optimal input state is the singlet:

$$|\psi\rangle = (U_{\text{id}} \otimes U_{\text{id}})(\mathbb{1} \otimes -i\sigma_y) |\Phi^+\rangle = (U_{\text{id}} \otimes U_{\text{id}}) |\Psi^-\rangle = e^{i\gamma} |\Psi^-\rangle. \quad (65)$$

This setup—initializing the singlet $|\Psi^-\rangle$ and rotating the TLSs oppositely—defines our positronium metrology protocol. Therefore, positronium metrology achieves the maximal QFI, $\mathcal{I}_\alpha^{(-)} = 4$, regardless of the direction $\hat{\mathbf{n}}$. Below, we prove that, if $\hat{\mathbf{n}}$ is unknown and one can consume only two units of space–time volume, to achieve the greatest possible QFI, one must (i) entangle two TLSs and (ii) effectively invert the field experienced by one TLS.

D. Effective unitary inversion, combined with entanglement, is the only strategy whose QFI achieves the greatest possible QFI attainable if $\hat{\mathbf{n}}$ is unknown and $v_{\text{st}} = 2$

Below, we prove that effective unitary inversion and entanglement enable the unique optimal sensing strategy available when (i) $\hat{\mathbf{n}}$ is unknown and (ii) the sensing can consume only $v_{\text{st}} = 2$ units of space–time volume. By

optimal, we mean that the strategy's QFI equals the greatest possible QFI achievable, $\mathcal{I}_\alpha = 4$. This equality does not depend on the rotation axis.

Two protocol structures incur space-time volumes $v_{\text{st}} = 2$, as explained in Suppl. Note I. First, one TLS may undergo two sequential applications of $U_\alpha = e^{-i\alpha H}$. In this scenario, the QFI attains its maximum value, 4, if the input state is an equal-weight superposition of the eigenstates of $H = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2$. These eigenstates depend on $\hat{\mathbf{n}}$, so this strategy is not deterministically achievable if $\hat{\mathbf{n}}$ is unknown. Second, two TLSs may experience the external field simultaneously, such that each undergoes U_α or $U_\alpha^\dagger$. We continue to analyze this scenario below. We show that the only optimal strategy meets two conditions: (i) The input state $|\psi\rangle$ is a singlet. (ii) One TLS undergoes U_α , while the other undergoes $U_\alpha^\dagger$.

To identify the optimal strategy, we begin consider a general pure two-TLS state:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \quad \text{wherein} \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1. \quad (66)$$

This state's T matrix depends on the coefficients' real ($\Re$) and imaginary ($\Im$) components as

$$T = \begin{pmatrix} 2\Re(ad^* + bc^*) & 2\Im(bc^* - ad^*) & 2\Re(ac^* - bd^*) \\ -2\Im(ad^* + bc^*) & 2\Re(bc^* - ad^*) & 2\Im(bd^* - ac^*) \\ 2\Re(ab^* - cd^*) & 2\Im(cd^* - ab^*) & |a|^2 - |b|^2 - |c|^2 + |d|^2 \end{pmatrix}. \quad (67)$$

The optimal $\hat{\mathbf{n}}$ -independent strategy implies, by Eq. (26), that $\mathcal{I}_\alpha^{(s)} = 2(1 + s\hat{\mathbf{n}}^T T \hat{\mathbf{n}}) = 4$, $\forall \hat{\mathbf{n}}$. This equation simplifies algebraically to $\hat{\mathbf{n}}^T T \hat{\mathbf{n}} = s$. Since this equation holds for all $\hat{\mathbf{n}}$ (by the strategy's independence of $\hat{\mathbf{n}}$), $T = s\mathbb{1}$. This condition is equivalent, by Eq. (67), to the system of equations

$$\begin{cases} |a|^2 - |b|^2 - |c|^2 + |d|^2 = s \\ 2\Re(bc^* + ad^*) = s \\ 2\Re(bc^* - ad^*) = s. \end{cases} \quad (68)$$

T has a unit determinant. Therefore, if T satisfies the equations above, its off-diagonal elements vanish. By the second and third equations, $\Re(ad^*) = 0$, and $\Re(bc^*) = s/2$. Furthermore, by the Cauchy-Schwarz inequality, $|\Re(bc^*)| \leq |b| \cdot |c|$. By the normalization condition in Eq. (66), $|b| \cdot |c| \leq \frac{1}{2}$. These conditions, with $s = \pm 1$, implies a chain of inequalities:

$$\frac{1}{2} = \left| \frac{s}{2} \right| = |\Re(bc^*)| \leq |b| \cdot |c| \leq \frac{1}{2}. \quad (69)$$

Since the leftmost and rightmost expressions equal each other, the two inequalities are saturated. To saturate the first inequality, b and c must have the same phase: $b = |b|e^{i\gamma}$, and $c = |c|e^{i\gamma}$. To saturate the second inequality, b and c must satisfy $|b| = |c| = 1/\sqrt{2}$. The normalization of $|\psi\rangle$ [Eq. (66)] then implies $a = d = 0$, which satisfies the earlier requirement $\Re(ad^*) = 0$ [Eq. (68)].

Consider substituting $|b| = |c| = 1/\sqrt{2}$ and $a = d = 0$ in the first equality in (68). The condition $s = -1$ results ($s = +1$ leads to a contradiction). Only the $s = -1$ state $|\psi\rangle$ —the singlet—satisfies the $\hat{\mathbf{n}}$ -independent-optimality condition $T = s\mathbb{1}$: $|\psi\rangle = \frac{e^{i\gamma}}{\sqrt{2}}(|10\rangle - |01\rangle)$.

E. Proof that positronium metrology achieves an FI of $\mathcal{I}_\alpha = 4$ while consuming a space-time volume $v_{\text{st}} = 2$

Here, we prove that positronium metrology attains an FI of $\mathcal{I}_\alpha = 4$ while consuming two units of space-time volume. This FI equals the corresponding QFI, as proved in Suppl. Note IB.

Positronium metrology involves three steps, as explained in Sec. V of the main text. First, we prepare a synthetic-positronium atom (a qubit q and an antiqubit $\bar{q}$) in a singlet, $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Second, we apply the unitary U_α to q and $U_\alpha^\dagger$ to $\bar{q}$. The joint state evolves to $|\psi_\alpha\rangle = (U_\alpha \otimes U_\alpha^\dagger)|\Psi^-\rangle = \frac{1}{\sqrt{2}}(e^{-i\alpha}|01\rangle - e^{i\alpha}|10\rangle)$. Finally, we projectively measure the POVM $\{|\Psi^-\rangle\langle\Psi^-|, \mathbb{1} - |\Psi^-\rangle\langle\Psi^-|\}$.

Having reviewed the protocol, we calculate the FI achievable with it. The measurement yields the $|\Psi^-\rangle\langle\Psi^-|$ outcome with a probability

$$P(|\Psi^-\rangle) = |\langle\Psi^-|\psi_\alpha\rangle|^2 = \cos^2(\alpha). \quad (70)$$

This probability serves as the $P_{j=1}$ in the FI formula $I_\alpha = \sum_j \frac{(\partial_\alpha P_j)^2}{P_j}$ (reviewed in Sec. I of the main text). The other probability is $P_2 = 1 - P_1$. We substitute for P_2 into the FI formula:

$$I_\alpha = \sum_j \frac{(\partial_\alpha P_j)^2}{P_j} = \frac{(\partial_\alpha P_1)^2}{P_1} + \frac{(\partial_\alpha P_2)^2}{P_2} = \frac{(\partial_\alpha P_1)^2}{P_1} + \frac{[\partial_\alpha(1 - P_1)]^2}{1 - P_1} = \frac{(\partial_\alpha P_1)^2}{P_1} + \frac{(\partial_\alpha P_1)^2}{1 - P_1} = \frac{(\partial_\alpha P_1)^2}{P_1(1 - P_1)}. \quad (71)$$

The remaining probability, $P_1 = P(|\Psi^-\rangle) = \cos^2(\alpha)$, has a derivative $\partial_\alpha P(|\Psi^-\rangle) = -2\cos(\alpha)\sin(\alpha)$. Substituting into the FI formula yields

$$I_\alpha = \frac{[\partial_\alpha P(|\Psi^-\rangle)]^2}{P(|\Psi^-\rangle)[1 - P(|\Psi^-\rangle)]} = \frac{4\cos^2(\alpha)\sin^2(\alpha)}{\cos^2(\alpha)\sin^2(\alpha)} = 4. \quad (72)$$

We proved in Suppl. Note IB that the QFI, too, equals 4. Applying positronium metrology, one can achieve an FI equal to the corresponding QFI (the greatest possible FI).

II. QFI ACHIEVABLE WITH SYNTHETIC ANTIMATTER AND NO ENTANGLEMENT

Positronium metrology exploits antimatter simulation and entanglement (Sec. II of the main text). We now isolate the entanglement's contribution to the metrological advantage. To do so, we evaluate the QFI attainable with (i) synthetic antimatter and (ii) the input state used in our separable-state experiment, as reported on in the main text. We calculate the QFI available to a metrologist who does not know the field orientation $\hat{\mathbf{n}}$ during the experiment.

First, we briefly review the separable protocol implemented experimentally. The two transmons are prepared in a product state: the qubit q is initialized in $|x+\rangle$; and the antiquubit $\bar{q}$, in $|z+\rangle$. (More generally, the two TLSs may begin in any two orthogonal pure states.) The external field applies $U_\alpha = e^{i\alpha H}$ to q and $U_\alpha^\dagger = e^{-i\alpha H}$ to $\bar{q}$. The generator has the form $H = (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})/2$, and $\hat{\mathbf{n}}$ denotes the unknown field direction. The qubit is measured with the POVM $\{|x+\rangle\langle x+|, |x-\rangle\langle x-|\}$, which has a probability $P(|x+\rangle)$ of yielding the $x+$ outcome. Simultaneously, the antiquubit is measured with $\{|z+\rangle\langle z+|, |z-\rangle\langle z-|\}$, which yields the $z+$ outcome with a probability $P(|z+\rangle)$. From these probabilities, one can estimate α .

In the main text, we assumed that the metrologist learns $\hat{\mathbf{n}}$ after the experiment concludes. This assumption simplified the analysis without affecting our main conclusion: entangled antimatter outperforms unentangled antimatter. Here, we operate under a more stringent assumption: the direction $\hat{\mathbf{n}}$ remains unknown at all times. The motivation is the need to compare the entanglement-free and positronium-metrology strategies fairly. One can use positronium metrology without knowing the external field's direction. Therefore, we assume that one does not know the direction when using unentangled synthetic antimatter.

A. Multiparameter quantum metrology

To assess the entanglement-free strategy available to a metrologist ignorant of $\hat{\mathbf{n}}$, we must review another metrological tool, the *quantum Fisher-information matrix* (QFIM). The metrologist aims to estimate the rotation angle α . However, their ignorance about $\hat{\mathbf{n}}$ naturally elevates the task to multiparameter estimation of not only α , but also the polar angle θ and azimuthal angle ϕ that defines $\hat{\mathbf{n}}$. In classical statistics, θ and ϕ act as nuisance parameters—variables that influence the estimation of α but are not themselves of interest [70]. To accommodate them in one's analysis, one extends the FI framework to the multiparameter framework, defining an effective FI that marginalizes the nuisance contributions. Similarly, we calculate an effective single-parameter QFI, $\mathcal{I}_{\alpha,\text{eff}}$, that captures the average precision with which one can estimate α without knowing the field direction.

In multiparameter quantum metrology, the QFIM quantifies a strategy's effectiveness [66, 71]. To introduce the QFIM, we define $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \dots, \vartheta_k)$ as the vector of k parameters to be estimated. Let $\rho_{\boldsymbol{\vartheta}}$ denote a state dependent on the vector. Associated with each parameter ϑ_i is an operator called the *symmetric logarithmic derivative* (SLD), L_i . It is defined implicitly through

$$\partial_{\vartheta_i} \rho_{\boldsymbol{\vartheta}} = \frac{1}{2} (\rho_{\boldsymbol{\vartheta}} L_i + L_i \rho_{\boldsymbol{\vartheta}}). \quad (73)$$

Using the SLDs, we can express the QFIM achievable with this state. The QFIM is a $k \times k$ positive-semidefinite matrix with the elements

$$\mathcal{M}_{ij}(\boldsymbol{\vartheta}) = \frac{1}{2} \text{Tr}(\rho_{\boldsymbol{\vartheta}} \{L_i, L_j\}), \quad \text{wherein } i, j = 1, 2, \dots, k. \quad (74)$$

The $\{A, B\} := AB + BA$ denotes the anticommutator of the operators A and B .

Having defined the QFIM, we review its relevant properties. The QFIM generalizes the single-parameter quantum Fisher information (QFI). If $k = 1$, $\mathcal{M}_{ii}$ reduces to the QFI with respect to ϑ_i . However, when one estimates multiple parameters simultaneously, the QFI does not quantify a strategy's effectiveness. Rather, a matrix-valued bound (the *quantum Cramér–Rao bound*) captures the limitations on simultaneous parameter estimation. To derive the quantum Cramér–Rao bound, we review estimators and covariance matrices. Consider a measurement that yields outcome k with a probability $p(k|\boldsymbol{\vartheta})$. An estimator is a function $\hat{\boldsymbol{\vartheta}}(k)$ that maps each k to a candidate estimate of $\boldsymbol{\vartheta}$. We focus on locally unbiased estimators, which satisfy the two conditions:

$$\sum_k [\boldsymbol{\vartheta}_i - \hat{\boldsymbol{\vartheta}}_i(k)] p(k|\boldsymbol{\vartheta}) = 0, \quad \text{and} \quad \sum_k \hat{\boldsymbol{\vartheta}}_i(k) \frac{\partial p(k|\boldsymbol{\vartheta})}{\partial \vartheta_j} = \delta_{ij}. \quad (75)$$

According to the first condition, the estimator tracks the parameter's true value faithfully, to first order around the point $\boldsymbol{\vartheta}$. The second constraint excludes pathological estimators. Examples include an estimator that returns a fixed value, irrespectively of the measurement outcome. We quantify an unbiased estimator's accuracy $\hat{\boldsymbol{\vartheta}}$ with covariance matrix,

$$\text{Cov}(\hat{\boldsymbol{\vartheta}}) := \sum_k \left[\hat{\boldsymbol{\vartheta}}(k) - \boldsymbol{\vartheta} \right] \left[\hat{\boldsymbol{\vartheta}}(k) - \boldsymbol{\vartheta} \right]^T p(k|\boldsymbol{\vartheta}). \quad (76)$$

To express the Cramér–Rao bound concisely, we introduce the Loewner (positive-semidefinite) ordering: if A and B denote Hermitian operators, then $A \succeq B$ means that $A - B$ is positive-semidefinite. In terms of this definition, the multiparameter quantum Cramér–Rao bound (QCRB) has the form

$$\text{Cov}(\hat{\boldsymbol{\vartheta}}) \succeq \mathcal{M}^{-1}(\boldsymbol{\vartheta}). \quad (77)$$

Let us apply this formalism to our setting. The QFIM for the three unknown parameters (α , θ , and ϕ) assumes the form

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{\alpha\alpha} & \mathcal{M}_{\alpha\theta} & \mathcal{M}_{\alpha\phi} \\ \mathcal{M}_{\theta\alpha} & \mathcal{M}_{\theta\theta} & \mathcal{M}_{\theta\phi} \\ \mathcal{M}_{\phi\alpha} & \mathcal{M}_{\phi\theta} & \mathcal{M}_{\phi\phi} \end{pmatrix}. \quad (78)$$

Define $\mathcal{M}_{\hat{\mathbf{n}}\hat{\mathbf{n}}}$ as the lower-right 2×2 block, associated with the nuisance parameters. Also, define $\mathcal{M}_{\alpha\hat{\mathbf{n}}}^T := (\mathcal{M}_{\alpha\theta}, \mathcal{M}_{\alpha\phi})$. This two-element vector quantifies the correlations between α , and $\hat{\mathbf{n}}$. In terms of these quantities, we can express the QFIM compactly:

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{\alpha\alpha} & \mathcal{M}_{\alpha\hat{\mathbf{n}}}^T \\ \mathcal{M}_{\hat{\mathbf{n}}\alpha} & \mathcal{M}_{\hat{\mathbf{n}}\hat{\mathbf{n}}} \end{pmatrix}. \quad (79)$$

Now, we bound the precision with which one can estimate α , without knowing $\hat{\mathbf{n}}$. We adapt the multiparameter QCRB [Eq. (77)] by invoking two properties of matrices. One is the Loewner ordering. The other is the non-negativity of every positive-semidefinite matrix's diagonal elements. From these ingredients, we derive $\text{Var}(\hat{\alpha}) \geq (\mathcal{M}^{-1})_{\alpha\alpha}$. We refine this bound using the Schur complement formula [70, 72]:

$$\text{Var}(\hat{\alpha}) \geq (\mathcal{M}^{-1})_{\alpha\alpha} = (\mathcal{M}_{\alpha\alpha}^{-1} - \mathcal{M}_{\alpha\hat{\mathbf{n}}}^T \mathcal{M}_{\hat{\mathbf{n}}\hat{\mathbf{n}}}^{-1} \mathcal{M}_{\hat{\mathbf{n}}\alpha})^{-1}. \quad (80)$$

The QFIM is positive-semidefinite, so the second term in Eq. (80) is non-negative. Therefore, we can achieve less precision in the presence of nuisance parameters than when performing single-parameter estimation. In the latter case, $\text{Var}(\hat{\alpha}) \geq \mathcal{M}_{\alpha\alpha}^{-1}$. The two bounds coincide if and only if $\mathcal{M}_{\alpha\hat{\mathbf{n}}} = 0$ —if and only if the QFIM is block-diagonal:

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{\alpha\alpha} & 0 \\ 0 & \mathcal{M}_{\hat{\mathbf{n}}\hat{\mathbf{n}}} \end{pmatrix} \quad \text{for all } (\alpha, \hat{\mathbf{n}}). \quad (81)$$

If the QFIM lacks this structure, we can define an effective QFI $\mathcal{I}_{\alpha, \text{eff}}$ for α . We average over a prior distribution for $\hat{\mathbf{n}}$:

$$\mathcal{I}_{\alpha, \text{eff}} := \langle (\mathcal{M}^{-1})_{\alpha\alpha} \rangle_{\hat{\mathbf{n}}} = \left\langle (\mathcal{M}_{\alpha\alpha}^{-1} - \mathcal{M}_{\alpha\hat{\mathbf{n}}}^T \mathcal{M}_{\hat{\mathbf{n}}\hat{\mathbf{n}}}^{-1} \mathcal{M}_{\hat{\mathbf{n}}\alpha})^{-1} \right\rangle_{\hat{\mathbf{n}}}. \quad (82)$$

If $\hat{\mathbf{n}}$ is unknown, a natural prior is the uniform distribution over the Bloch sphere.

In summary, $\mathcal{I}_{\alpha, \text{eff}}$ is useful under two simultaneous conditions: (i) The QFIM does not exhibit the block-diagonal structure in Eq. (81). (ii) The metrologist estimating α does not know the values of θ and ϕ . The entanglement-free experiment depicted in Fig. 3(c) meets condition (i). Condition (ii) is an assumption introduced in this supplementary note so that we can fairly compare the entanglement-free strategy with positronium metrology, in which $\hat{\mathbf{n}}$ is unknown.

B. Calculation of the effective QFI achievable without knowledge of the field direction

We now apply the QFIM's properties to assess the entanglement-free antimatter strategy applicable without knowledge of $\hat{\mathbf{n}}$. As detailed earlier, the entanglement-free protocol begins with the qubit-antiquubit pair in the state $|x+\rangle_{\text{q}} \otimes |z+\rangle_{\bar{\text{q}}}$. The external field evolves the state to

$$|\psi_{\alpha}\rangle = U_{\alpha} |x+\rangle_{\text{q}} \otimes U_{\alpha}^{\dagger} |z+\rangle_{\bar{\text{q}}}. \quad (83)$$

Every product state's QFIM is additive [66]:

$$\mathcal{M}(|\psi_{\alpha}\rangle) = \mathcal{M}(U_{\alpha}|x+\rangle_{\text{q}}) + \mathcal{M}(U_{\alpha}^{\dagger}|z+\rangle_{\bar{\text{q}}}). \quad (84)$$

We can calculate $\mathcal{M}(U_{\alpha}|x+\rangle_{\text{q}})$ and $\mathcal{M}(U_{\alpha}^{\dagger}|z+\rangle_{\bar{\text{q}}})$ straightforwardly. They reveal that $\mathcal{M}(|\psi_{\alpha}\rangle)$ has nonzero off-diagonal elements. They encode correlations between α and the nuisance parameters. These correlations confirm the usefulness of the effective QFI, $\mathcal{I}_{\alpha,\text{eff}}$ [Eq. (82)].

To calculate $\mathcal{I}_{\alpha,\text{eff}}$, we calculate $(\mathcal{M}^{-1})_{\alpha\alpha}$. Its value follows from Eq. (84), which we can evaluate straightforwardly:

$$(\mathcal{M}^{-1})_{\alpha\alpha} = \frac{1}{8} [7 + \cos(2\theta) + 2 \cos(2\phi) \sin^2(\theta)]. \quad (85)$$

Because $\hat{\mathbf{n}}$ is unknown, we average $(\mathcal{M}^{-1})_{\alpha\alpha}$ uniformly over the possible field directions $\hat{\mathbf{n}}$. Denote this average by $\langle \cdot \rangle_{\hat{\mathbf{n}}}$. To evaluate it, we integrate each side of Eq. (85) over the unit sphere:

$$\langle (\mathcal{M}^{-1})_{\alpha\alpha} \rangle_{\hat{\mathbf{n}}} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} (\mathcal{M}^{-1})_{\alpha\alpha} \sin(\theta) d\theta d\phi = \frac{5}{6}. \quad (86)$$

By the quantum Cramér-Rao bound [Eq. (77)],

$$\langle \text{Var}(\alpha) \rangle_{\hat{\mathbf{n}}} \geq \langle (\mathcal{M}^{-1})_{\alpha\alpha} \rangle_{\hat{\mathbf{n}}} = \frac{5}{6}. \quad (87)$$

The effective QFI follows from Eq. (82):

$$\mathcal{I}_{\alpha,\text{eff}} = [\langle (\mathcal{M}^{-1})_{\alpha\alpha} \rangle_{\hat{\mathbf{n}}}]^{-1} = \left(\frac{5}{6}\right)^{-1} = \frac{6}{5} = 1.2. \quad (88)$$

This effective QFI is slightly lower than the optimal QFI achievable without entanglement but with a space-time volume $v_{\text{st}} = 2$, $4/3 \approx 1.33$. One can achieve this optimal QFI using a tensor product of two copies of the optimal input state used when the probe consists of just one qubit. This state is a mixture of three mutually unbiased states (Suppl. Note C.4 of [72]). One should expect our experiment's entanglement-free strategy to perform suboptimally, since the QFIM (84) is not diagonal.

Finally, we compare the entanglement-free synthetic-antimatter strategy with positronium metrology. Positronium metrology achieves a QFIM the form (81), and $\mathcal{I}_{\alpha} = 4$. By Eq. (88), entanglement enhances the achievable precision significantly.

III. EXPERIMENTAL REALIZATION OF QUBITS AND ANTIQUBITS

This supplementary note reviews the two strategies we employ to realize antiquubits. As discussed in the main text, we apply Z gates to $\bar{\text{q}}$ to effectively reverse the rotation induced by the field's x - and y -components. Supplementary Note IV B details the Z -gate implementation. We induce the field's z -component using an AC Stark shift with a frequency chosen such that $\delta_{\text{q}} = -\delta_{\bar{\text{q}}}$. This choice causes q and $\bar{\text{q}}$ to rotate about the z -axis oppositely.

To begin, we calculate the AC Stark shifts conferred upon q and $\bar{\text{q}}$ by an off-resonant drive. Recall that the qubit has a frequency ω_{q} . Denote by α_{q} the qubit's anharmonicity, the difference (gap between second and third energy levels)–(gap between first and second energy levels). Consider subjecting the qubit to an oscillating external field of amplitude Ω_{s} and frequency ω_{s} . The field has a detuning $\Delta_{\text{qs}} := \omega_{\text{q}} - \omega_{\text{s}}$. The qubit acquires an AC Stark shift [73]

$$\delta_{\text{q}} = \frac{\alpha_{\text{q}} \Omega_{\text{s}}^2}{2\Delta_{\text{qs}}(\alpha_{\text{q}} + \Delta_{\text{qs}})}. \quad (89)$$

	$\omega_{q,\bar{q},c}/(2\pi)$ (GHz)	$\alpha/(2\pi)$ (MHz)	Resonator (GHz)	T_1 (μ s)	T_2^* (μ s)
Qubit	4.16748	-146.916	6.69	28	35
Antiqubit	4.27398	-144.658	6.88	17	22
Coupler	5.24975	-152.384	7.08	14	16

TABLE I. **Device parameters.**

The antiqubit obeys an analogous result. q and $\bar{q}$ have different frequencies, so a “magic” drive frequency induces equal-magnitude, opposite-sign Stark shifts on the transmons. Every magic frequency depends on the device parameters. Table I specifies the parameters’ values. Two drive frequencies enable $\delta_q = -\delta_{\bar{q}}$; one is 4.19742 GHz. During the experimental calibration, we found that $\delta_q = -\delta_{\bar{q}}$ at 4.176998 GHz. The inferred magic frequency differs slightly from the prediction because q and $\bar{q}$ experience different field amplitudes: $\bar{q}$ experiences a field strength that is ≈ 1.78 times larger. In Fig. 2(f), the tick marks along the x -axis show the Ω_s values (the drive amplitudes experienced by q) inferred from Eq. (89). The inferred magic frequency is scarcely detuned from the qubit’s transition frequency: $\Delta_{qs}/(2\pi) = -9.52$ MHz. Therefore, the oscillating field does not only induce δ_q and $\delta_{\bar{q}}$. Also, the field drives off-resonant rotations about the qubit’s x - or y -axis, depending on the drive’s phase. These imperfections imprint in Fig. 3(d), in the gray disks inferred after rotations about the z -axis. The gray disks approximately form a curve that wobbles more than the figure’s other approximate curves.

We infer the AC Stark shift by performing simultaneous Ramsey measurements on q and $\bar{q}$. To do so, we rotate each transmon through an angle $\pi/2$ about the $\hat{x}$ -axis. Then, we apply the Stark tone to the readout driveline for an amount of time that varies from batch of trials to batch of trials. Each transmon then undergoes another $\pi/2$ rotation about a different axis (relative to the first $\pi/2$ rotation) to produce a synthetic detuning. Finally, we read out the transmons’ states. Figure 2(e) displays the measured frequency shifts, as well as theoretical predictions. Using the measured data, we identified a magic frequency. To obtain the results in Fig. 3, we used frequency shifts $|\delta_j| = \Omega_x = \Omega_y = 2\pi(2.13 \text{ MHz})$, wherein $j \in \{q, \bar{q}\}$. To rotate each transmon through an angle $\alpha \in [0, 2\pi]$, we drove them for amounts of time $\in [0, 470]$ ns.

IV. EXPERIMENTAL SETUP

The experiments were performed on a superconducting device with three transmon circuits. Table I specifies the device parameters. We denote the qubit transmon by q ; the antiqubit, by $\bar{q}$; and the coupler, by c . The qubit’s frequency is fixed, whereas c and $\bar{q}$ are flux-tunable. We operate c and $\bar{q}$ with net zero flux threading their superconducting-quantum-interference-device (SQUID) loops. Each transmon has a ground state $|g\rangle$, first excited state $|e\rangle$, and second excited state $|f\rangle$. Dispersively coupled readout resonators read out all the transmons’ states. The device was fabricated in the SQUILL foundry at MIT Lincoln Laboratory [74]. We mounted the device on a dilution refrigerator’s mixing-chamber stage, with attenuated coaxial lines for implementing qubit drives and flux biases. Reference [75] provides details about the overall setup at the hardware level.

A. Dispersive readout

The qubit and antiqubit couple to their respective readout resonators [76] dispersively. We can thereby perform simultaneous high-fidelity single-shot readouts [77]. Our readout scheme is heterodyne: we multiplex the readout signal by simultaneously sending in two pulses that have different frequencies. We separate the two readouts in the frequency domain for processing. To implement low-noise amplification, we use a traveling-wave parametric amplifier based on SNAILs (Superconducting Nonlinear Asymmetric Inductive elements) [78]. To increase the signal-to-noise ratio, we apply π -pulses that promote q and $\bar{q}$ from their $|e\rangle$ to the $|f\rangle$ levels. We optimize the readout signal-to-noise ratio over readout amplitude, frequency, and amplifier-bias settings, using a gradient-free optimization algorithm [79]. We construct the multicomponent-pulse-integration envelopes via linear-discriminant analysis and principal-component analysis. Ultimately, we achieve a qubit-readout fidelity of 97.8 % and an antiqubit-readout fidelity of 95.0 %, using a random forest classifier [80]. After this calibration, we correct all tomography results for the finite readout fidelities, using the iterative Bayesian-update correction method [81].

B. Single-qubit rotations

We use four types of single-qubit rotations in this project:

- Single-qubit $\pi/2$ rotations: We use $\pi/2$ rotations about the x - and y -axes in quantum state tomography and in the arbitrary-axis rotations detailed below. Each $\pi/2$ rotation lasts for 44 ns. We construct pulse envelopes from cosine waveforms. As a function of the time t (expressed in ns), the pulse waveform has the form $\propto [\cos(2\pi t/(44 \text{ ns}) - \pi) + 1]$. We estimate these gates' fidelity to be 98.8 %.
- Single-qubit π rotations: We implement these rotations in the same manner as the $\pi/2$ rotations. However, each pulse lasts for 88 ns.
- Single-qubit π rotations in the $\{|e\rangle, |f\rangle\}$ manifold: These rotations are implemented in the same manner as the $\{|g\rangle, |e\rangle\}$ rotations described in the previous two bullet points. The $\pi/2$ gates last for 32 ns each; and the π gates, 64 ns.
- Single-qubit rotations about the z -axis, through arbitrary angles α : Denote by $R(\beta, \phi)$ a rotation through an angle β about the $(\cos(\phi)\hat{x} + \sin(\phi)\hat{y})$ -axis. The corresponding rotation operator is represented by the matrix, relative to the Z -eigenbasis,

$$R(\beta, \phi) = \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) & -ie^{-i\phi}\sin\left(\frac{\beta}{2}\right) \\ -ie^{i\phi}\sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right) \end{pmatrix}. \quad (90)$$

In terms of this rotation, we define the *physical* rotation through an angle α about the z -axis,

$$R_z(\alpha) = R\left(\pi, \frac{\alpha}{2}\right) R(\pi, 0). \quad (91)$$

$$(92)$$

C. Parametric gates

We implement parametric gates by modulating the coupler's frequency: define $\Delta_{\bar{q}q} \equiv \omega_{\bar{q}} - \omega_q$. We apply a microwave tone to c 's fast flux line at a frequency $\Delta_{\bar{q}q}/2$, modulating c 's energy gaps. Denote the modulation depth by α . The coupler's time-dependent frequency is $\tilde{\omega}_c(t) = \omega_c - \alpha \cos^2(\Delta_{\bar{q}q}t/2)$. During this modulation, q and $\bar{q}$ come into parametric resonance. When calibrated, the parametric resonance induces qubit–antiqubit coupling at a rate of 54.0 MHz. At this rate, an $\sqrt{i\text{SWAP}}$ gate lasts 104 ns.

We estimate the gate's fidelity using quantum state tomography. To do so, we measure the expectation values of 9 two-Pauli products: $\langle \sigma_q \sigma_{\bar{q}} \rangle$. We use maximum-likelihood estimation to infer the qubit–antiqubit density matrix's components [82].

We implement as follows the protocol shown in Fig. 3(a). To prepare an entangled state, we perform one $\sqrt{i\text{SWAP}}$ gate. Then, we projectively measure whether the system is in a singlet, $|\Psi^-\rangle$. That is, we measure $\{\Pi_0, \mathbb{1} - \Pi_0\}$ (as in the main text, $\Pi_0 := |\Psi^-\rangle\langle\Psi^-|$). To measure this POVM, we apply a second $\sqrt{i\text{SWAP}}$ gate, which maps the singlet to $|g\rangle_q |e\rangle_{\bar{q}}$. The qubits have different energies, so they accumulate different relative phases between the two $\sqrt{i\text{SWAP}}$ gates. To remove this phase difference, we apply another physical rotation to the antiqubit before the second $\sqrt{i\text{SWAP}}$ gate. We implement this phase correction with the physical rotation (91) described above.